Symbolic Dynamics and Unpredictability Defined by Right Adjointness

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Abstract. It is shown that a category-theoretic definition of chaotic system applies not only to the Smale horseshoe, a prototypical example, but also to Conway's "Life." Symbolic dynamics of the "Dining Philosophers" relational system is computed. A category composed of stochastic matrices is defined and its elementary properties are studied. A categorical variant of symbolic dynamics is applied to a finite stochastic process. Using point-wise Kan extension formulas, conditions ensuring existence of certain representations between categories of dynamic systems are proved.

Keywords. category theory; symbolic dynamics; chaos; discrete time systems; cellular automaton

1. INTRODUCTION

In 1900 David Hilbert posed, with optimism for the future of his science, twenty-three problems which still challenge mathematicians. The sixth involves a mathematical treatment of the axioms of physics and was influenced by his success in geometry. Contributing to the solution of this problem, Walter Noll laid down axioms in 1958 for a general theory describing materials which make up natural bodies and had revised this work by 1972 [15]. Lawvere's 1967 lectures (published in 1979 [7]) outlined his "program to (3) axiomatize the foundations of continuum mechanics in the spirit of Walter Noll on the basis of (2) a direct axiomatization of the essence of differential topology using results and methods of the French work in algebraic geometry ... this requires (1) axiomatic study of categories of smooth sets, similar to the toposes of Grothendieck, since the most natural form of (2) is incompatible with usual set theory."

This paper is intended to contribute to a program of developing general concepts and methods for the modeling and solution of scientific problems. The language of category theory is employed because it facilitates precise comparisons between diverse types of structures. Characterizing idioms occurring across classifications of dynamic systems may be useful in relating distinct models of fluid flows, such as kinetic and continuum descriptions [16].

Lambek and Scott [6] emphasize six succinct, categorical slogans in their logical treatise and attribute most of these to Lawvere. In [8] and [9], Lawvere reveals that the slogans also pertain to investigations of dynamic systems. Such structures changing over time often occur as objects of functor categories and may be classified according to the manner in which time advances and the nature of the state spaces and transition rules. For a monoid $\mathcal{M}$, an object of $\mathcal{C}^{\mathcal{M}}$ is an autonomous system having time-independent state space of type determined by the category $\mathcal{C}$. An object of $\mathcal{C}^{\mathcal{P}}$ with $\mathcal{P}$ an ordered set may have variable state spaces and transition rules. Periodic points, orbits, and symbolic dynamics are important concepts arising as adjoints to simple functors. This work presents applications of such categorical constructions to some well-known dynamic systems.

2. PRELIMINARIES

The foundations in [11] are employed. The adjective *small* describes: a set which is a member of the fixed universe U; a category which has a small set of morphisms; a functor mapping one small category into another. $|\mathcal{J}|$ and $\{\mathcal{J}\}$ respectively denote the sets of objects and morphisms of a small category, $\mathcal{J}$. Functors which are full, embeddings, and faithful [3] are respectively displayed by $\longrightarrow\!\!\!\triangleright$, $\longmapsto$, and $\Vdash\!\!\longrightarrow$. A morphism $\longmapsto$ is a monomorphism while $\longrightarrow\!\!\!\triangleright$ is an epimorphism. All diagrams are commutative unless indicated otherwise. $\mathbb{N} = \{0, 1, \ldots\}$ is the set of natural numbers and $\mathbb{R}^+$ denotes the nonnegative reals. A <u>category action</u> is a functor $X : \mathcal{J} \to \mathcal{C}$ with $\mathcal{J}$ small. For fixed $\mathcal{C}$, a small functor $F : \mathcal{J} \to \mathcal{J}'$ induces a representation $\mathcal{C}^F : \mathcal{C}^{\mathcal{J}'} \to \mathcal{C}^{\mathcal{J}}$ between categories of actions via $X \mapsto X \circ F$ and $\tau \mapsto \tau \cdot id_F$ where $\cdot$ is horizontal composition of natural transformations. Left and right adjoints to such functors are respectively denoted Lan_F and Ran_F [11]. Small functors arising in the context of dynamic systems include:

$$|\mathcal{J}| \xrightarrow{\eta_{\mathcal{J}}} \mathcal{J} \xrightarrow{!} 1$$

with $\eta_{\mathcal{J}}$ inclusion, $\mathcal{J}$ a monoid or ordered set as special cases, and with 1 a one-morphism category;

$$\begin{array}{ccccc}
(\mathbb{N}, \leq) & \xrightarrow{\delta} & (\mathbb{N}, 0, +) & \xrightarrow{\natural_n} & (\mathbb{N}_n, 0, +) \\
\downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} & & \\
(\mathbb{R}^+, \leq) & \xrightarrow[\delta']{} & (\mathbb{R}^+, 0, +) & \xrightarrow[\natural_x]{} & (\mathbb{R}^+_x, 0, +)
\end{array}$$

with ι and ι' inclusions, δ and δ' differences: $(x, y) \mapsto y - x$; $\natural_n$ projection onto the positive integers mod(n), and $\natural_x$ also a projection. These induce representations:

$$\mathcal{C}^{|\mathcal{J}|} \xleftarrow{\mathcal{C}^{\eta_{\mathcal{J}}}} \mathcal{C}^{\mathcal{J}} \xleftarrow{T_{\mathcal{J}}} \mathcal{C}$$

with $T_{\mathcal{J}} = \mathcal{C}^! \circ \Delta$ and with $\Delta : \mathcal{C} \to \mathcal{C}^1$ an isomorphism; and

$$\begin{array}{ccccc}
\mathcal{C}^{(\mathbb{N}, \leq)} & \xleftarrow{\mathcal{C}^{\delta}} & \mathcal{C}^{(\mathbb{N}, 0, +)} & \xleftarrow{\mathcal{C}^{\natural_n}} & \mathcal{C}^{(\mathbb{N}_n, 0, +)} \\
\uparrow{\scriptstyle \mathcal{C}^{\iota'}} & & \uparrow{\scriptstyle \mathcal{C}^{\iota}} & & \\
\mathcal{C}^{(\mathbb{R}^+, \leq)} & \xleftarrow[\mathcal{C}^{\delta'}]{} & \mathcal{C}^{(\mathbb{R}^+, 0, +)} & \xleftarrow[\mathcal{C}^{\natural_x}]{} & \mathcal{C}^{(\mathbb{R}^+_x, 0, +)}.
\end{array}$$

Lawvere and others interpret these and their adjoints [1], [8]–[10]. $T_{\mathcal{J}}$ constructs a system with trivial dynamics from a space. $\mathcal{C}^{\eta_{\mathcal{J}}}$ transforms a dynamic system into a mere $|\mathcal{J}|$-indexed set of spaces. $\Delta^{-1} \circ \mathcal{C}^{\eta_{\mathcal{M}}}$ gives the underlying space of a monoid action and is also denoted ev_* with $*$ the lone $\mathcal{M}$-object. $\mathcal{C}^{\natural_n}$ and $\mathcal{C}^{\natural_x}$ both place periodic systems in contexts of non-periodic ones while $\mathcal{C}^{\delta}$ and $\mathcal{C}^{\delta'}$ assert that autonomous systems are special non-autonomous ones. Applying $\mathcal{C}^{\iota}$ or $\mathcal{C}^{\iota'}$ is the process of observing a continuous-time system at discrete intervals.

3. EXISTENCE OF ADJOINTS

Adjoints to these representations yield important concepts [1], [8]–[10].

Theorem 1: *A right adjoint exists:*

- *i. for $T_{\mathcal{J}}$ iff $\mathcal{C}$ admits $\mathcal{J}$-limits;*
- *ii. for $\mathcal{C}^{\eta_{\mathcal{J}}}$ if $\mathcal{C}$ has products indexed by nonempty subsets of $\{\mathcal{J}\}$;*
- *iii. for $\mathcal{C}^{\delta}$ and $\mathcal{C}^{\natural_n}$ if $\mathcal{C}$ has equalizers and countable products;*
- *iv. for $\mathcal{C}^{\delta'}$, $\mathcal{C}^{\iota}$, $\mathcal{C}^{\iota'}$, and $\mathcal{C}^{\natural_x}$ if $\mathcal{C}$ has equalizers and $\aleph$-products where $\aleph = 2^{\aleph_0}$ is the cardinality of $\mathbb{R}$;*

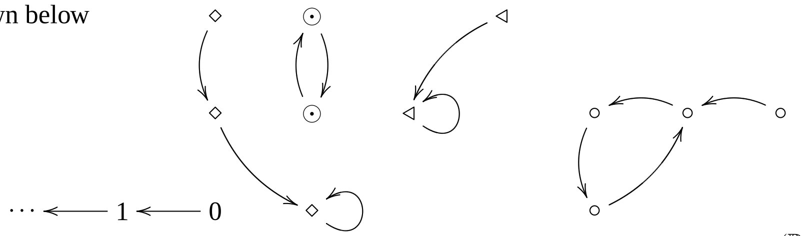
A left adjoint exists:

- *i.$^\circ$ for $T_{\mathcal{J}}$ iff $\mathcal{C}$ admits $\mathcal{J}$-colimits;*
- *ii.$^\circ$ for $\mathcal{C}^{\eta_{\mathcal{J}}}$ if $\mathcal{C}$ has coproducts indexed by nonempty subsets of $\{\mathcal{J}\}$;*
- *iii.$^\circ$ for $\mathcal{C}^{\delta}$ and $\mathcal{C}^{\natural_n}$ if $\mathcal{C}$ has coequalizers and countable coproducts;*
- *iv.$^\circ$ for $\mathcal{C}^{\delta'}$, $\mathcal{C}^{\iota}$, $\mathcal{C}^{\iota'}$, and $\mathcal{C}^{\natural_x}$ if $\mathcal{C}$ has coequalizers and $\aleph$-coproducts.*

Proof: i)$^\circ$ is Theorem 1 from page 248 of [11]. In general, a limit (co- limit) formula for right (left) Kan extensions [11], characterization of the appropriate comma categories [18], and the fact that limits (colimits) are constructed from equalizers and products (coequalizers and coproducts) yield each result. $\mathcal{C}^{\eta_{\mathcal{J}}}$, for example, has a right adjoint if a right Kan extension along $\eta_{\mathcal{J}}$ exists for each $Y : |\mathcal{J}| \to \mathcal{C}$. Given Y, this extension exists if $\mathcal{C}$ admits a limit of $J \downarrow \eta_{\mathcal{J}} \xrightarrow{p} |\mathcal{J}| \xrightarrow{Y} \mathcal{C}$ for each $J \in |\mathcal{J}|$, where $J \downarrow \eta_{\mathcal{J}}$ is the category of arrows from J to $\eta_{\mathcal{J}}$ and p is the projection. $J \downarrow \eta_{\mathcal{J}}$, however, is discrete: its objects are the $\mathcal{J}$-morphisms with domain J.

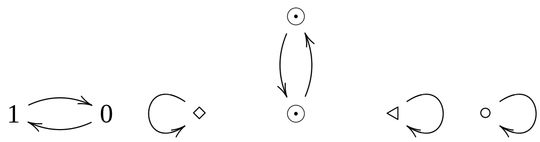
∎

For a monoid $\mathcal{M}$, right and left adjoints of $T_{\mathcal{M}}$ respectively give fixed points and orbits [1], [8]. $\mathrm{Ran}_{T_{\mathcal{M}}}$ maps the iterator $X : (\mathbb{N}, 0, +) \to \mathbf{Set}$ shown below



to $\{\diamond, \triangleleft\}$ while a left adjoint gives the set $\{0, \diamond, \odot, \triangleleft, \circ\}$. Applying a left adjoint of $T : \mathbf{Set} \to \mathbf{Set}^{(\mathbb{R}^+, 0, +)}$ to the flow on $\mathbb{R}$ induced by $\dot{x} = \alpha x \left(1 - \frac{x}{\beta}\right)$ also produces a five-point set. $\mathrm{Ran}_{\natural_n}$ maps $(\mathbb{N}, 0, +) \to \mathbf{Set}$ to its period-n part while $\mathrm{Lan}_{\natural_n}$ forces such a system into a period-n mold by "collapsing states" [8]. $\mathrm{Ran}_{\natural_2}$ and $\mathrm{Lan}_{\natural_2}$, for example, respectively map the iterator X to

and

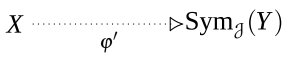


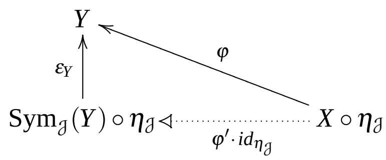
4. SYMBOLIC DYNAMICS AND CHAOS

Intuitively, a dynamic system is chaotic if continued observations do not aid in predicting its behavior. Precise definitions traditionally involve topology in an essential way through denseness of an orbit or topological transitivity [17]. Definitions in this section are due to Lawvere [8]–[10]. A <u>measurement</u> on a dynamic system $X : \mathcal{J} \to \mathcal{C}$ is a natural transformation $\varphi : X \circ \eta_{\mathcal{J}} \to Y$ for some $Y : |\mathcal{J}| \to \mathcal{C}$. φ assigns a morphism $\varphi_J : X(J) \to Y(J)$ on the space $X(J)$ at each 'time' $J \in |\mathcal{J}|$. Being an arrow from $\mathcal{C}^{\eta_{\mathcal{J}}}$ to Y, it induces a representation between dynamic systems whenever $\mathcal{C}^{\eta_{\mathcal{J}}}$ has a right adjoint,

$$\mathrm{Sym}_{\mathcal{J}} : \mathcal{C}^{|\mathcal{J}|} \to \mathcal{C}^{\mathcal{J}}.$$

This construction enjoys the universal property expressed by





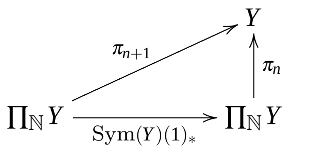
with ε the counit and φ' the unique, induced arrow. Such a right adjoint deserves the name *symbolic dynamics* because, if $\mathcal{C} = \mathbf{Set}$ and $\mathcal{J}$ is the monoid $(\mathbb{N}, 0, +)$ then

$$\mathrm{Sym}_{\mathcal{J}}(Y)(*) = Y^{\mathbb{N}}, \qquad \big[\mathrm{Sym}_{\mathcal{J}}(Y)(1)\big](\nu) = \nu \circ \sigma, \qquad \text{and} \qquad \varepsilon_Y(\nu) = \nu(0)$$

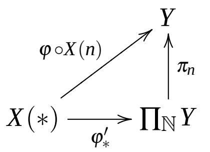
where $\sigma : \mathbb{N} \to \mathbb{N}$ is successor. Moreover,

$$\mathrm{Sym}_{\mathcal{J}}(k)_*(\nu) = k \circ \nu \qquad \text{and} \qquad \varphi'_*(x)(n) = \varphi\big(X(n)(x)\big) = \varphi\big(X(1)^n(x)\big)$$

for $k \in \mathbf{Set}(Y, Y')$, $x \in X(*)$, and $n \in \mathbb{N}$. For any $\mathcal{C}$ having (some choice of) countable products, $\mathrm{Sym}_{\mathcal{J}}(Y)(*) = \prod_{\mathbb{N}} Y$ and ε_Y is the projection π_0, while $\mathrm{Sym}_{\mathcal{J}}(Y)(1)$ and φ'_* are respectively the unique morphisms for which



and



for all $n \in \mathbb{N}$. If $\mathcal{C} = \mathbf{Top}$, $\prod_{\mathbb{N}} Y$ has the compact-open topology with $\mathbb{N}$ a discrete space. In **Rel**, the category composed of [3] binary relations, products and coproducts coincide, changing the nature of symbolic dynamics. Let categories $\mathcal{J}$ and $\mathcal{C}$ be such that $\mathcal{C}^{\eta_{\mathcal{J}}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}^{|\mathcal{J}|}$ has a right adjoint, $\mathrm{Sym}_{\mathcal{J}}$. A measurement $\varphi : X \circ \eta_{\mathcal{J}} \to Y$ on $X : \mathcal{J} \to \mathcal{C}$ is <u>chaotic</u>

if Y is not a terminator and the induced $\varphi' : X \to \mathrm{Sym}_{\mathcal{J}}(Y)$ is an epimorphism. This property of *measurements* gives the property of *systems*. A dynamic system $X : \mathcal{J} \to \mathcal{C}$ is <u>chaotic</u> if it has a subobject $X' \mapsto X$ on which there exists a chaotic measurement $X' \circ \eta_{\mathcal{J}} \to Y$.

5. HORSESHOE MAPS

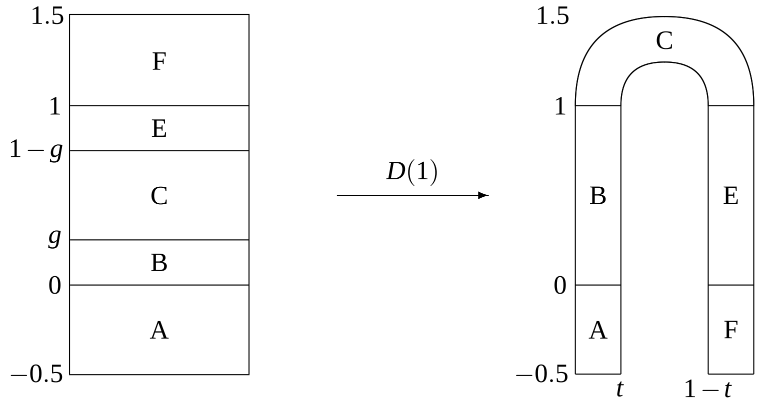
Motivated by problems in ordinary differential equations, Smale showed in 1963 that most diffeomorphisms of a manifold having dimension greater than one possess infinitely many periodic points and a richly-structured invariant set [12]. The horseshoe maps described below, for example, have such properties. They are the prototypical examples of chaotic systems [17]. For each pair (t, g) of parameters with $0 < t < \frac{1}{2}$ and $0 < g < \frac{1}{2}$, let $D : (\mathbb{N}, 0, +) \to \mathbf{Top}$ have

$$D(*) = \left\{ (x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ -\frac{1}{2} \le y \le \frac{3}{2} \right\}$$

equipped with the subspace topology. Define $D(1) : D(*) \to D(*)$ by

$$D(1)(x,y) \ = \ \begin{cases} (tx,\ y) & \text{if } y < 0; \\ \left(tx,\ \frac{1}{g}y\right) & \text{if } 0 \le y \le g; \\ \left(\frac{1}{2} - \left(\frac{1-2tx}{2}\right)\cos\left(\frac{\pi(y-g)}{1-2g}\right),\ 1 + \left(\frac{1-2tx}{2}\right)\sin\left(\frac{\pi(y-g)}{1-2g}\right)\right) & \text{if } g < y < 1-g; \\ \left(1-tx,\ \frac{1}{g}(1-y)\right) & \text{if } 1-g \le y \le 1; \\ (1-tx, 1-y) & \text{if } y > 1. \end{cases}$$

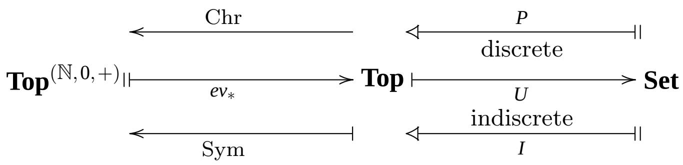
The figure below illustrates this action.



Fix any positive integer n. That $\left(\frac{1}{1+t^n}, \frac{g^{n-1}}{1+g^n}\right)$ has an orbit of period n suggests the complexity of repeated foldings.

Theorem 2: *A horseshoe map is chaotic in the categorical sense.*

Proof: The proof delivers the forward part of the invariant set in [17] and is a rearrangement of that argument into a categorical mold. It uses



where left adjoints appear above their right adjoints and commutativity is not asserted. P assigns power set topologies to sets while the topology on $I(Y)$ is $\{\phi, Y\}$. It is sufficient to find a chaotic measurement $A(*) \to P(2)$ on a subobject A of D with $2 = \{0, 1\}$. Let $S = \{0, 1, r\}$ and define $\phi : U(D(*)) \to S$ by

$$\phi(x,y) \ = \ \begin{cases} 0 & \text{if } x < 1/2 \text{ and } 0 \le y \le 1; \\ 1 & \text{if } x \ge 1/2 \text{ and } 0 \le y \le 1; \\ r & \text{otherwise.} \end{cases}$$

$\mathbf{Top}^{(\mathbb{N},0,+)}$	**Top**	**Set**

Column headings above indicate categories in which the structures below live:

D	$D(*)$	$U(D(*))$

for example. The function

$$U(D(*)) \xrightarrow{\phi} S$$

induces a continuous map

$$D(*) \longrightarrow I(S)$$

by right adjointness. This induces

$D \longrightarrow \mathrm{Sym}(I(S))$

again by right adjointness. Injective functions are monomorphisms in **Top** so

$$\begin{array}{ccc} & & P(2) \\ & & \downarrow \\ D(*) & \longrightarrow & I(S) \end{array}$$

gives a subobject. Right adjoints preserve limits, hence

$$\begin{array}{ccc} & & \mathrm{Sym}(P(2)) \\ & & \downarrow \\ D & \longrightarrow & \mathrm{Sym}(I(S)) \end{array}$$

is a subobject. Since $\mathbf{Top}^{(\mathbb{N},0,+)}$ is complete, there is a pullback

$$\begin{array}{ccc} A & \xrightarrow[\varphi']{} & \mathrm{Sym}(P(2)) \\ \downarrow & & \downarrow \\ D & \longrightarrow & \mathrm{Sym}(I(S)). \end{array}$$

Evaluation is a right adjoint so it preserves this pullback:

$$\begin{array}{ccc} A(*) & \xrightarrow[\varphi'_*]{} & P(2)^{\mathbb{N}} \\ \downarrow & & \downarrow \\ D(*) & \longrightarrow & I(S)^{\mathbb{N}}. \end{array}$$

Commutativity of this last diagram is the assertion that A is a subobject of D remaining in the unit square for all time. Pullback asserts that it is the maximal such. $\varphi' : A \to \mathrm{Sym}(P(2))$ is induced by $\mathrm{ev}_* \dashv \mathrm{Sym}$ and $\varphi = \mathrm{ev}_0 \circ \varphi'_*$ with ev_0 sequence-evaluation at $n = 0$. The following lemmas establish surjectivity of φ'. Fix ε satisfying $t < \varepsilon < 1/2$. For real numbers u and v define $\alpha_{u,v} = \{(x,y) \mid 0 \le x \le \varepsilon,\ u \le y \le v\}$ and $\beta_{u,v} = \{(x,y) \mid 1-\varepsilon \le x \le 1,\ u \le y \le v\}$. For finite binary sequences $x_0, x_1, \ldots, x_n$ define sets $V_{x_0,x_1,\ldots,x_n}$ by $V_0 = \alpha_{0,1}$ and $V_1 = \beta_{0,1}$ for $n = 0$ and $V_{x_0,x_1,\ldots,x_n} = V_{x_0} \cap D(1)^{-1}(V_{x_1,\ldots,x_n})$ for $n \ge 1$.

Lemma 1: *The equalities below hold whenever* $0 \le u \le v \le 1$.

$$V_0 \cap D(1)^{-1}(\alpha_{u,v}) = \alpha_{gu,gv} \qquad V_1 \cap D(1)^{-1}(\alpha_{u,v}) = \beta_{gu,g,v}$$

$$V_0 \cap D(1)^{-1}(\beta_{u,v}) = \alpha_{1-gv,1-gu} \qquad V_1 \cap D(1)^{-1}(\beta_{u,v}) = \beta_{1-gv,1-gu}$$

Proof: $D(1)(x,y) = \big(f(x,y), h(x,y)\big)$ implies

$$\begin{aligned} D(1)^{-1}(\alpha_{u,v}) &= \{(x,y) \mid D(1)(x,y) \in \alpha_{u,v}\} \\ &= \{(x,y) \mid 0 \le f(x,y) \le \varepsilon, u \le h(x,y) \le v\} \\ &= \{(x,y) \mid 0 \le y \le g, 0 \le f(x,y) \le \varepsilon, u \le h(x,y) \le v\} \\ &= \{(x,y) \mid 0 \le y \le g, 0 \le tx \le \varepsilon, u \le y/g \le v\} \\ &= \{(x,y) \mid 0 \le y \le g, gu \le y \le gv\} \\ &= \{(x,y) \mid gu \le y \le gv\} \end{aligned}$$

and $V_0 \cap D(1)^{-1}(\alpha_{u,v}) = \{(x,y) \mid 0 \le x \le \varepsilon, gu \le y \le gv\} = \alpha_{gu,gv}$. Proofs of the remaining equalities are similar. ∎

Lemma 2: *For each finite binary sequence $x_0, x_1, \ldots, x_n$, there are numbers u and v with $0 \le u \le v \le 1$ such that*

$$V_{x_0,x_1,\ldots,x_n} = \begin{cases} \alpha_{u,v} & \text{if } x_0 = 0; \\ \beta_{u,v} & \text{if } x_0 = 1. \end{cases}$$

It follows that $V_{x_0,x_1,\ldots,x_n}$ is compact and nonempty.

Proof: Use induction on binary sequence length. For $n = 0$, $V_0 = \alpha_{0,1}$ and $V_1 = \beta_{0,1}$ by definition. Assume that $V_{x_0,x_1,\ldots,x_n}$ has the form described for any binary sequence x_0, x_1, …, x_n with $n \le N$. Given $x_0, x_1, \ldots, x_{N+1}$, the induction hypothesis implies that there are numbers u and v satisfying $0 \le u \le v \le 1$ for which $V_{x_1,\ldots,x_{N+1}} = \alpha_{u,v}$ or $V_{x_1,\ldots,x_{N+1}} = \beta_{u,v}$. In the former case, $V_{x_0,x_1,\ldots,x_n} = V_{x_0} \cap D(1)^{-1}(\alpha_{u,v})$ and in the latter $V_{x_0,x_1,\ldots,x_n} = V_{x_0} \cap D(1)^{-1}(\beta_{u,v})$. By Lemma 1, these imply existence of numbers u' and v' satisfying $0 \le u' \le v' \le 1$ for which $V_{x_0,x_1,\ldots,x_{N+1}} = \alpha_{u',v'}$ or $V_{x_0,x_1,\ldots,x_{N+1}} = \beta_{u',v'}$. ∎

Lemma 3: *Points of $V_{x_0,x_1,\ldots,x_n}$ realize the measurements sequence x_0, x_1, …, x_n. That is: $(x,y) \in V_{x_0,x_1,\ldots,x_n}$ implies*

$$\varphi(x,y) = x_0, \qquad \varphi \circ D(1)(x,y) = x_1, \qquad \ldots, \qquad \varphi \circ D(n)(x,y) = x_n.$$

Proof: Use induction on binary sequence length. For $n = 0$: $(x,y) \in V_0$ implies $\varphi(x,y) = 0$; $(x,y) \in V_1$ implies $\varphi(x,y) = 1$. Assume that $V_{x_0,x_1,\ldots,x_n}$ has the property described for any binary sequence x_0, x_1, …, x_n with $n \le N$. $(x,y) \in V_{x_0,x_1,\ldots,x_{N+1}} = V_{x_0} \cap D(1)^{-1}(V_{x_1,\ldots,x_{N+1}})$ implies $(x,y) \in V_{x_0}$ so $\varphi(x,y) = x_0$. Since

$$\begin{aligned} D(1)(V_{x_0,x_1,\ldots,x_{N+1}}) &= D(1)\Big(V_{x_0} \cap D(1)^{-1}(V_{x_1,\ldots,x_{N+1}})\Big) &&\subset D(1)(V_{x_0}) \cap D(1)\big(D(1)^{-1}(V_{x_1,\ldots,x_{N+1}})\big) \\ &\subset D(1)(V_{x_0}) \cap V_{x_1,\ldots,x_{N+1}} &&\subset V_{x_1,\ldots,x_{N+1}}\,, \end{aligned}$$

$(x,y) \in V_{x_0,x_1,\ldots,x_{N+1}}$ implies that $D(1)(x,y)$ induces x_1, …, x_{N+1}, hence, (x,y) induces x_0, x_1, …, x_{N+1}. ∎

Lemma 4: *x_0, x_1, x_2, … a binary sequence implies*

$$V_{x_0} \supset V_{x_0,x_1} \supset V_{x_0,x_1,x_2} \supset \cdots$$

a descending chain of nonempty, compact subspaces of $D(*)$.

Proof: Each set is compact and nonempty by Lemma 2. Induction on binary sequence length justifies the containment claim. $V_{x_0,x_1} = V_{x_0} \cap D(1)^{-1}(V_{x_1})$. Assume that

$$V_{x_0,x_1,\ldots,x_n} = V_{x_0,x_1,\ldots,x_{n-1}} \cap (D(1)^{-1})^n(V_{x_n})$$

for any binary sequence x_0, x_1, …, x_n with $n \le N$. Then

$$\begin{aligned} V_{x_0,x_1,\ldots,x_{N+1}} &= V_{x_0} \cap D(1)^{-1}(V_{x_1,\ldots,x_{N+1}}) &&= V_{x_0} \cap D(1)^{-1}\Big(V_{x_1,\ldots,x_N} \cap (D(1)^{-1})^N(V_{x_{N+1}})\Big) \\ &= V_{x_0} \cap D(1)^{-1}(V_{x_1,\ldots,x_N}) \cap (D(1)^{-1})^{N+1}(V_{x_{N+1}}) &&= V_{x_0,x_1,\ldots,x_N} \cap (D(1)^{-1})^{N+1}(V_{x_{N+1}}). \end{aligned}$$

∎

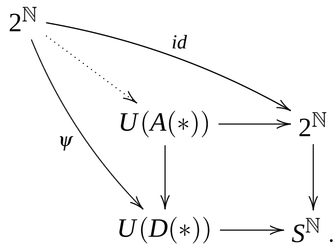
Lemma 5: *There is a function $\psi : 2^{\mathbb{N}} \to U(D(*))$ for which*

$$\begin{array}{ccc} 2^{\mathbb{N}} & \xrightarrow{\;id\;} & 2^{\mathbb{N}} \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ U(D(*)) & \longrightarrow & S^{\mathbb{N}}. \end{array}$$

Proof: For x_0, x_1, …, sets in the sequence $V_{x_0} \supset V_{x_0,x_1} \supset \cdots$ are compact, nonempty subsets of $D(*) \subset \mathbb{R}^2$ by Lemma 4 so $\bigcap_{n=0}^{\infty} V_{x_0,\ldots,x_n}$ is compact and nonempty as well. The axiom of choice implies existence of ψ. ∎

Lemma 6: $A \longrightarrow\!\!\triangleright Sym(P(2))$ *is an epimorphism.*

Proof: $\begin{array}{ccc} A(*) & \longrightarrow & P(2)^{\mathbb{N}} \\ \downarrow & & \downarrow \\ D(*) & \longrightarrow & I(S)^{\mathbb{N}} \end{array}$ a pullback in **Top** implies $\begin{array}{ccc} U(A(*)) & \longrightarrow & 2^{\mathbb{N}} \\ \downarrow & & \downarrow \\ U(D(*)) & \longrightarrow & S^{\mathbb{N}} \end{array}$ a pullback. Lemma 5 and the definition of pullback imply existence of a unique $2^{\mathbb{N}} \to U(A(*))$ for which:



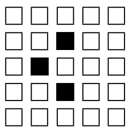
∎

6. "LIFE"

John H. Conway's game "Life" is a simple model of birth, death, and emergence in living systems [4]. Points of $\mathbb{Z}^2$ are *cells* which may be either *live* or *dead*. States of all cells give a function $f : \mathbb{Z}^2 \to 2$ with

$$f(x,y) \;=\; \begin{cases} 1 & \text{if } (x,y) \text{ is live;} \\ 0 & \text{if } (x,y) \text{ is dead.} \end{cases}$$

Figures such as



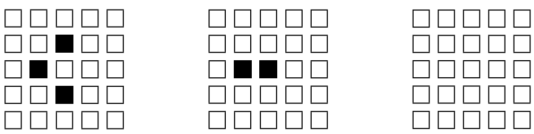
with ■ live and □ dead, depict collections of cells which evolve simultaneously and in discrete time intervals: if

$$\begin{array}{ccc} 5 & 6 & 7 \\ 3 & \circ & 4 \\ 0 & 1 & 2 \end{array}$$

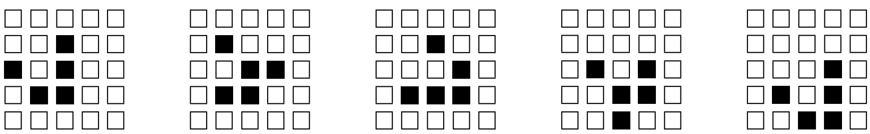
denotes a cell and its eight neighbors and ℓ is the number of live adjacent cells then

$$\text{the subsequent state of cell } \circ = \begin{cases} \text{the current state of } \circ & \text{if } \ell = 2; \\ 1 & \text{if } \ell = 3; \\ 0 & \text{otherwise.} \end{cases}$$

Sequences of figures



illustrate evolution of a few cells. A *glider* is a useful structure which evolves through four shapes during its motion:



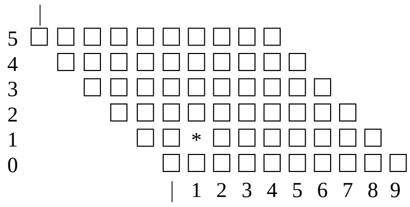
"Life" is a functor $L : (\mathbb{N}, 0, +) \to$ **Set** with $L(*)$ the set of functions $[\mathbb{Z}^2, 2]$. Its evolution rule determines a formula for $L(1) : [\mathbb{Z}^2, 2] \to [\mathbb{Z}^2, 2]$ and was designed to make the population behavior unpredictable.

Theorem 3: *"Life" is chaotic.*

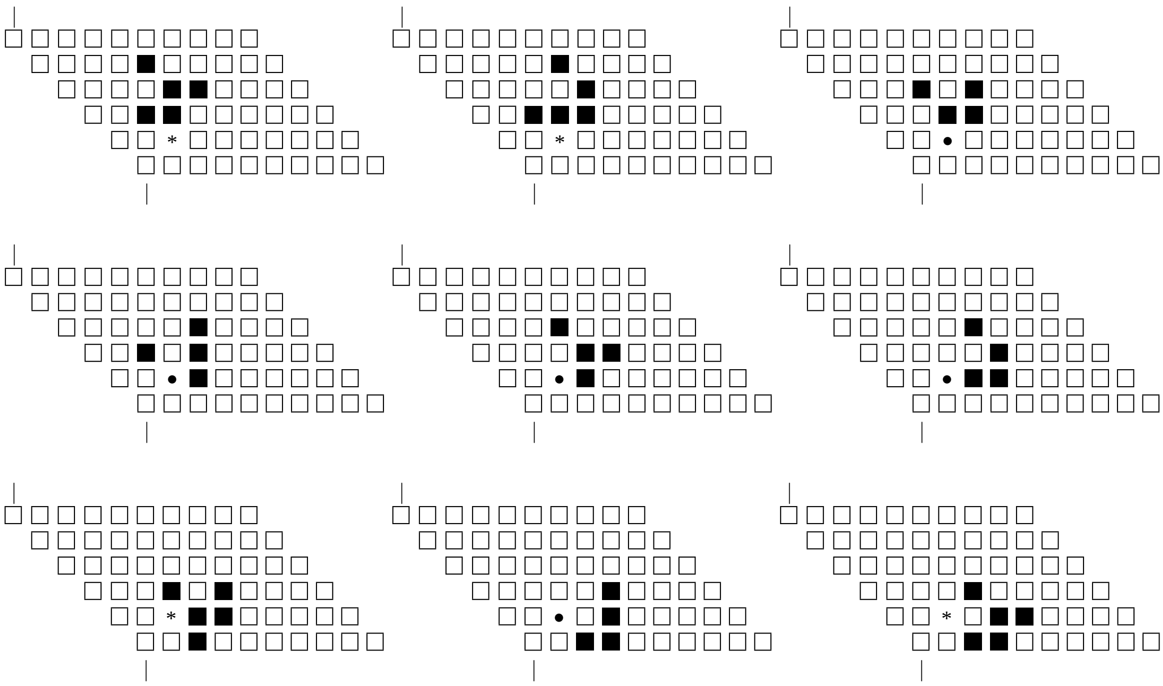
Proof: It is sufficient to show that there is a measurement $\varphi : [\mathbb{Z}^2, 2] \to 2$ for which the induced $\varphi'_* : [\mathbb{Z}^2, 2] \to 2^{\mathbb{N}}$ is a surjection. For $n \in \mathbb{N}$,

$$R_n = \left\{ (x,y) \in \mathbb{Z}^2 \mid 10\,n - x \leq y < 10\,(n+1) - x \right\}$$

is a *strip*. The point $z_n = (10\,n + 1,\, 1)$ of R_n is *special*. The figure



shows part of the strip R_0 with the special cell z_0 denoted $*$. A live special cell is indicated by $\bullet$. Motion of a glider through a strip



induces states $\cdots, 0, 1, 1, 1, 1, 0, 1, 0, \cdots$ at the special cell and does not interfere with gliders passing through special cells in other strips. Gliders may be positioned in groupings of adjacent strips to produce measurements sequences of the form

$$\widehat{0} = 1, 0, \cdots \qquad \widehat{1} = 0, 1, 0, \cdots \qquad \widehat{2} = 0, 0, 1, 0, \cdots, \qquad \text{etc.}$$

Collections of such groupings produce arbitrary sequences. The measurement φ is defined inductively using the special cells in finite groupings of strips. L_n is the number of strips in grouping n. Let

$$L_0 = 1, \quad L_1 = 1, \quad L_2 = 2, \quad L_3 = 3, \quad L_4 = 3, \quad \text{and} \quad L_n = 1 + L_{n-2} + L_{n-3} + L_{n-4} + L_{n-5} \quad \text{if } n \geq 5.$$

Define $\text{start}_0 = 0$ and, for $n \geq 1$, let $\text{start}_n = \sum_{i=0}^{n-1} L_i$. Given $n \geq 0$ let

$$\varphi_n(f) = \left[\sum_{j=0}^{L_n - 1} f\left(z_{(j+\text{start}_n)}\right) \right] \text{mod}(2)$$

for $f \in [\mathbb{Z}^2, 2]$. Then

$$\varphi(f) = \max\{\varphi_n(f) \mid n \in \mathbb{N}\}$$

defines a function $\varphi : [\mathbb{Z}^2, 2] \to 2$, parts of which are tabulated below.

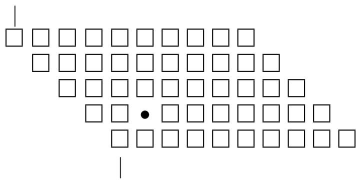
n	L_n	$\varphi_n(f)$
0	1	$f(1,1)$
1	1	$f(11,1)$
2	2	$f(21,1)+f(31,1) \quad \mathrm{mod}(2)$
3	3	$f(41,1)+f(51,1)+f(61,1) \quad \mathrm{mod}(2)$
4	3	$f(71,1)+f(81,1)+f(91,1) \quad \mathrm{mod}(2)$
5	8	$f(101,1)+f(111,1)+\cdots+f(171,1) \quad \mathrm{mod}(2)$
6	10	$f(181,1)+f(191,1)+\cdots+f(271,1) \quad \mathrm{mod}(2)$.

A measurements sequence $x_0, x_1, x_2, \cdots$ is realized by an initial state constructed inductively as follows:

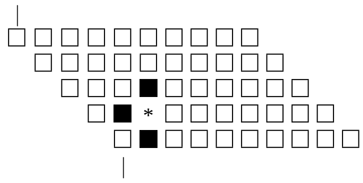
$$\text{fill grouping } n \text{ of strips with pattern} = \begin{cases} \text{all dead cells} & \text{if } x_n = 0; \\ \widehat{n} & \text{if } x_n = 1; \end{cases}$$

where the first five patterns are displayed in

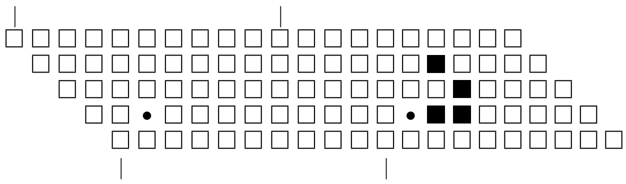
$\widehat{0} =$



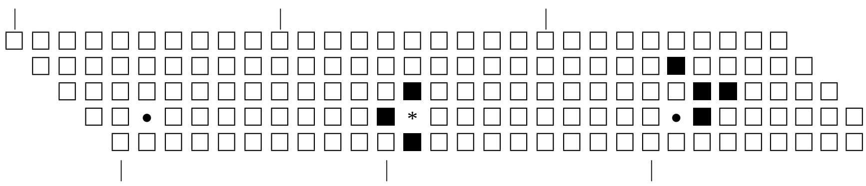
$\widehat{1} =$



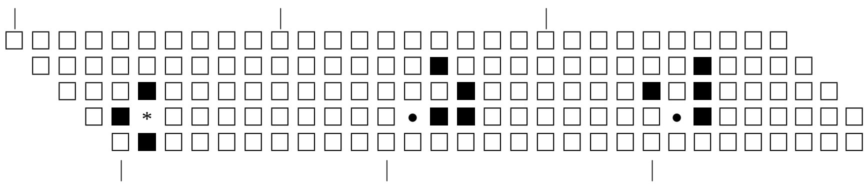
$\widehat{2} =$



$\widehat{3} =$



$\widehat{4} =$



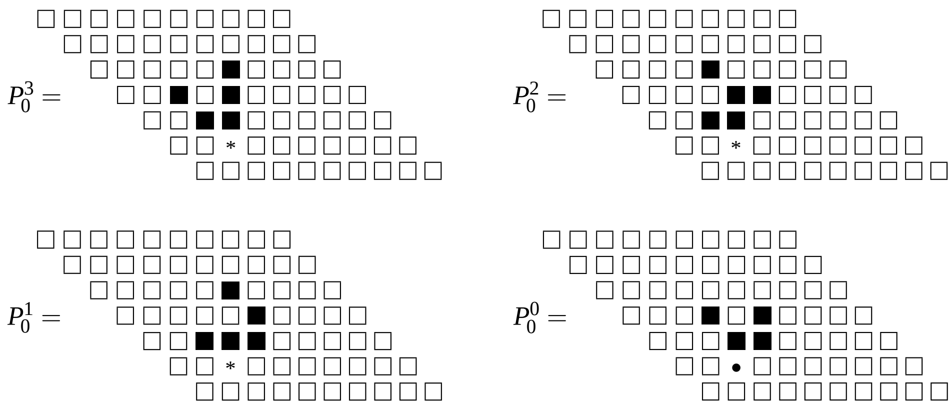
and the patterns for $n \geq 5$ are indicated by

$$\widehat{n} = \widehat{n-5}\,\widehat{n-4}\,\widehat{n-3}\,\widehat{n-2}\;\; P^{(n-5)\,\mathrm{mod}(4)}_{[[(n-5)/4]]}$$

defined to mean that grouping n of strips is filled with: a grouping of L_{n-5} strips containing the pattern $\widehat{n-5}$; followed by a grouping of L_{n-4} strips containing the pattern $\widehat{n-4}$; followed by a grouping of L_{n-3} strips containing the pattern $\widehat{n-3}$; followed by a grouping of L_{n-2} strips containing the pattern $\widehat{n-2}$; followed by a single strip with pattern

$$P^{(n-5)\,\mathrm{mod}(4)}_{[[(n-5)/4]]}$$

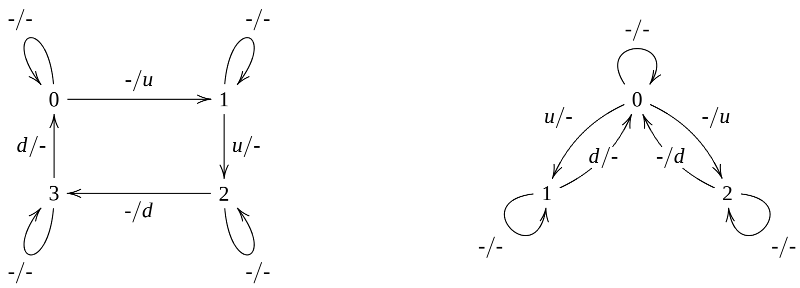
where $[[\cdot]]$ is the *floor* function ($[[1.3]] = 1$, for example). Patterns P_0^j are shown below.



and pattern P_m^j is obtained by shifting P_0^j up and to the left by m cells. The position of this last glider in $\widehat{n}$ is chosen so that it induces the measurements sequence $0, \cdots, 0, 1, 1, 1, 1, 0, 1, 0, \cdots$ in its strip with the final 1 occurring at time n. The other parts of $\widehat{n}$ mask the first four 1s in the computation of $\varphi_n(f)$. ∎

7. "DINING PHILOSOPHERS"

"Dining philosophers" refers to a simple, distributed, resource-allocation problem. A group of individuals is seated for a meal at a round table. A chopstick is placed between each pair of diners who must eat according to the following protocol: first pick up a chopstick with the right hand; next with the left; after eating, put down the right chopstick first; finally the left. At each stage a diner may rest or wait while others are active. Two or more diners may act simultaneously. The graph below left illustrates states and actions of a diner. A diner holding no chopsticks is in state 0. To eat, one must be in state 2.



States and transitions of a chopstick are shown above right. These two labeled graphs are from [13] where they are 1-cells in the bicategory **Span**(**Graph**). In that work, coordination between a chopstick and an adjacent diner is modeled by composition of 1-cells and the entire system is a large, labeled graph. No nontrivial transitions are allowed if each philosopher has a chopstick in his or her right hand. The challenge to the group is avoiding deadlock. A goal of modeling such systems is characterizing behaviors of the aggregate using descriptions of components. A system of two diners and two chopsticks has eight states.

The protocol is the relation between states tabulated below. The state at the top of a column may transition to the state labeling a row iff the row-column location is filled with the symbol $\circ$.

	$\bullet\!=\!\bullet$	$\bullet\,\_\,\overline{\bullet}$	$\bullet\quad\overline{\underline{\bullet}}$	$\bullet\,\overline{\ }\,\underline{\bullet}$	$\underline{\bullet}\,\overline{\ }\,\bullet$	$\overline{\underline{\bullet}}\quad\bullet$	$\overline{\bullet}\,\_\,\bullet$	$\underline{\bullet}\quad\overline{\bullet}$
$\bullet\!=\!\bullet$	$\circ$			$\circ$			$\circ$	
$\bullet\,\_\,\overline{\bullet}$	$\circ$	$\circ$						
$\bullet\quad\overline{\underline{\bullet}}$		$\circ$	$\circ$					
$\bullet\,\overline{\ }\,\underline{\bullet}$			$\circ$	$\circ$				
$\underline{\bullet}\,\overline{\ }\,\bullet$	$\circ$				$\circ$			
$\overline{\underline{\bullet}}\quad\bullet$					$\circ$	$\circ$		
$\overline{\bullet}\,\_\,\bullet$						$\circ$	$\circ$	
$\underline{\bullet}\quad\overline{\bullet}$	$\circ$	$\circ$			$\circ$			$\circ$

"Dining philosophers" is a functor $X : (\mathbb{N}, 0, +) \to \mathbf{Rel}$. $X(*)$ is the set of admissible states and $X(1) : X(*) \to X(*)$ is the relation defined by the protocol: $(x, x') \in X(1)$ iff the system may transition from x to x'. A system of two diners may be equipped with a measurement $\varphi \in \mathbf{Rel}(X(*), \{D, N\})$ indicating the condition of being in or out of deadlock. Symbolic dynamics gives the sequence of sets of measurements accessible from each initial state:

	0	1	2	3	
$\bullet\!=\!\bullet$	$\{N\}$	$\{D,N\}$	$\{D,N\}$	$\{D,N\}$	$\ldots$
$\bullet\,\_\,\overline{\bullet}$	$\{N\}$	$\{D,N\}$	$\{D,N\}$	$\{D,N\}$	$\ldots$
$\bullet\quad\overline{\underline{\bullet}}$	$\{N\}$	$\{N\}$	$\{N\}$	$\{D,N\}$	$\ldots$
$\bullet\,\overline{\ }\,\underline{\bullet}$	$\{N\}$	$\{N\}$	$\{D,N\}$	$\{D,N\}$	$\ldots$
$\underline{\bullet}\,\overline{\ }\,\bullet$	$\{N\}$	$\{D,N\}$	$\{D,N\}$	$\{D,N\}$	$\ldots$
$\overline{\underline{\bullet}}\quad\bullet$	$\{N\}$	$\{N\}$	$\{N\}$	$\{D,N\}$	$\ldots$
$\overline{\bullet}\,\_\,\bullet$	$\{N\}$	$\{N\}$	$\{D,N\}$	$\{D,N\}$	$\ldots$
$\underline{\bullet}\quad\overline{\bullet}$	$\{D\}$	$\{D\}$	$\{D\}$	$\{D\}$	$\ldots$

It is a relation $X(*) \xrightarrow{\varphi'_*} \prod_{\mathbb{N}}\{D,N\}$ with $\prod_{\mathbb{N}}\{D,N\} \cong \mathbb{N} \times \{D,N\}$ as sets since products and coproducts coincide in **Rel**. Of particular interest are states $\bullet\!=\!\bullet$, $\bullet\,\_\,\overline{\bullet}$, and $\underline{\bullet}\,\overline{\ }\,\bullet$ in which deadlock is imminent.

8. THE CATEGORY COMPOSED OF STOCHASTIC MATRICES

A real $n \times m$ matrix is <u>stochastic</u> if each entry is nonnegative and each column sum is 1. The category **StM** has positive integers as objects, stochastic matrices as morphisms, identity matrices as identity morphisms and matrix multiplication as composition.

Theorem 4: **StM** *has finite, nonempty coproducts.*

Proof: The category $\mathbf{Mes}_\Pi$ composed of transition kernels described in [5] and defined by Lawvere in an unpublished manuscript (see also [2]) is the Kleisli category of a particular monad on **Mes**, the category of measurable spaces. $F \dashv U$ is the associated adjunction. Powersets and trivial algebras give the other adjunctions in

$$\mathbf{Set} \; \underset{I}{\overset{P}{\rightleftarrows}} \; \mathbf{Mes} \; \underset{U}{\overset{F}{\rightleftarrows}} \; \mathbf{Mes}_\Pi .$$

Under the colimit-preserving functor $F \circ P$, finite sets take images in a subcategory of $\mathbf{Mes}_\Pi$ which is isomorphic to **StM** [18]. Unraveling formulas gives a coproduct $m \xrightarrow[\lambda]{} m+n \xleftarrow[\lambda']{} n$ in **StM** with

$$\lambda_{r,c} = \begin{cases} 1 & \text{if } r = c \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \lambda'_{r,k} = \begin{cases} 1 & \text{if } r = k+m \\ 0 & \text{otherwise} \end{cases}$$

for $1 \le r \le m+n$, $1 \le c \le m$, and $1 \le k \le n$. ∎

Theorem 5: **StM** *has finite, weak products. It does not admit all weak pullbacks.*
Proof: 1 is a terminator [3]. For objects m and n,

$$m \xleftarrow[\pi]{} mn \xrightarrow[\pi']{} n$$

is a weak product with

$$\pi_{r,c} = \begin{cases} 1 & \text{if } r = \left[\left[c - \frac{1}{n}\right]\right] + 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } 1 \le r \le m \text{ and } 1 \le c \le mn$$

$$\pi'_{r,c} = \begin{cases} 1 & \text{if } c \equiv r \bmod(n) \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } 1 \le r \le n \text{ and } 1 \le c \le mn$$

where $[[\cdot]]$ denotes the floor function. Given

$$\begin{array}{ccccc} & & p & & \\ & \swarrow^{A} & & \searrow^{B} & \\ m & \xleftarrow[\pi]{} & mn & \xrightarrow[\pi']{} & n. \end{array}$$

a matrix $(A,B) : p \to mn$ factoring through both projections has

$$(A,B)_{r,c} = A_{\sigma_m^{[[(r-1)/n]]}(1),c}\, B_{\sigma_n^{r-1}(1),c}$$

for $1 \le r \le mn$ and $1 \le c \le p$. σ_γ is the γ-cycle $(1\ 2\ \cdots\ \gamma)$ (e.g. $\sigma_5(1) = 2$, $\sigma_5^2(1) = 3$, and $\sigma_5^5(1) = \sigma_5^0(1) = 1$). This weak product construction is functorial **StM** $\times$ **StM** $\to$ **StM**. Counterexamples establish non-uniqueness of the factorization $p \to mn$ in the weak product construction and that **StM** does not have all weak pullbacks [18]. ∎

9. SYMBOLIC DYNAMICS OF A STOCHASTIC PROCESS

Peterson [14] introduces the following situation, and attributes it to Ehrenfests, to illustrate the Poincaré recurrence theorem. Three marbles are placed in two boxes, A and B. Each minute, one marble is randomly selected and moved from its current box to the other box. States of the system make up the set

$$\{AAA,\ AAB,\ ABA,\ BAA,\ ABB,\ BAB,\ BBA,\ BBB\}$$

with ABB the state having marble 1 in box A and the others in B. Assuming no bias in marble selection, each entry in

	AAA	*AAB*	*ABA*	*BAA*	*ABB*	*BAB*	*BBA*	*BBB*
AAA :	0	1/3	1/3	1/3	0	0	0	0
AAB :	1/3	0	0	0	1/3	1/3	0	0
ABA :	1/3	0	0	0	1/3	0	1/3	0
BAA :	1/3	0	0	0	0	1/3	1/3	0
ABB :	0	1/3	1/3	0	0	0	0	1/3
BAB :	0	1/3	0	1/3	0	0	0	1/3
BBA :	0	0	1/3	1/3	0	0	0	1/3
BBB :	0	0	0	0	1/3	1/3	1/3	0

gives the probability of a transition from the state labeling its column to the state labeling its row. A single transition is a functor X from the poset $\mathcal{P}_2 = 0 \xrightarrow{(0,1)} 1$ to **StM** with $X(0) = X(1) = 8$ and $X(0,1)$ given by the table above. Measuring the number of marbles in box A is the morphism $\varphi \in \textbf{StM}(8,4)$ tabulated atop the next page.

	AAA	*AAB*	*ABA*	*BAA*	*ABB*	*BAB*	*BBA*	*BBB*
0 :	0	0	0	0	0	0	0	1
1 :	0	0	0	0	1	1	1	0
2 :	0	1	1	1	0	0	0	0
3 :	1	0	0	0	0	0	0	0

Although **StM** has only weak products, not products, the Kan extension formula for symbolic dynamics of poset actions (Theorem 1(ii)) may be followed to construct a morphism induced by φ:

$$(8 \longrightarrow 8) \underset{\varphi'}{\dashrightarrow} (16 \xrightarrow{\pi'} 4) \qquad \begin{array}{ccc} (4,4) & \xleftarrow{(\varphi,\varphi)} & \\ \uparrow{\scriptstyle (\pi, id)} & & \\ (16,4) & \underset{\varphi'\circ\eta_{\mathcal{P}_2}}{\dashleftarrow} & (8,8). \end{array}$$

Symbolic dynamics gives information about probabilities of measurements sequences. If the system were initially in state AAB, for example. The likelihood of obtaining measurement 2 before transferring a marble, then measurement 1 after a transfer is the product of: the probability ($= 1$) that the initial measurement will be 2, the probability that the measurement will be 1 if the system starts in state AAB and one marble is transferred. These likelihoods are listed below and constitute a morphism φ' induced by (φ, φ).

	AAA	*AAB*	*ABA*	*BAA*	*ABB*	*BAB*	*BBA*	*BBB*
01 :	0	0	0	0	0	0	0	1
10 :	0	0	0	0	1/3	1/3	1/3	0
12 :	0	0	0	0	2/3	2/3	2/3	0
21 :	0	2/3	2/3	2/3	0	0	0	0
23 :	0	1/3	1/3	1/3	0	0	0	0
32 :	1	0	0	0	0	0	0	0

Each entry is the probability of obtaining the row-labeling measurements sequence when the system is initially in the column-labeling state. Successive transitions give a functor from $\mathcal{P}_3 = \; 0 \xrightarrow{(0,1)} 1 \xrightarrow{(1,2)} 2$ to **StM**. The induced φ' is shown in:

	AAA	*AAB*	*ABA*	*BAA*	*ABB*	*BAB*	*BBA*	*BBB*
010 :	0	0	0	0	0	0	0	1/3
012 :	0	0	0	0	0	0	0	2/3
101 :	0	0	0	0	7/27	7/27	7/27	0
103 :	0	0	0	0	2/27	2/27	2/27	0
121 :	0	0	0	0	14/27	14/27	14/27	0
123 :	0	0	0	0	4/27	4/27	4/27	0
210 :	0	4/27	4/27	4/27	0	0	0	0
212 :	0	14/27	14/27	14/27	0	0	0	0
230 :	0	2/27	2/27	2/27	0	0	0	0
232 :	0	7/27	7/27	7/27	0	0	0	0
321 :	2/3	0	0	0	0	0	0	0
323 :	1/3	0	0	0	0	0	0	0

The entry with column label AAB and row label 230, however, is not the probability of obtaining measurements sequence 2-3-0 for a system initially in state AAB. It is the product of: the probability that the initial measurement will be 2 if the system starts in state AAB, the probability that the measurement will be 3 if the system starts in state AAB and one marble is transferred, the probability that the measurement will be 0 if the system starts in state AAB and two marbles are transfered. The measurements sequence 2-3-0 is impossible since only one marble is transferred at a time. Actual likelihoods of measurements sequences are tabulated below. These values were not obtained from the construction in Theorem 1(ii) modified for use with weak products.

	AAA	*AAB*	*ABA*	*BAA*	*ABB*	*BAB*	*BBA*	*BBB*
010 :	0	0	0	0	0	0	0	$1/3$
012 :	0	0	0	0	0	0	0	$2/3$
121 :	0	0	0	0	$4/9$	$4/9$	$4/9$	0
123 :	0	0	0	0	$2/9$	$2/9$	$2/9$	0
101 :	0	0	0	0	$1/3$	$1/3$	$1/3$	0
210 :	0	$2/9$	$2/9$	$2/9$	0	0	0	0
212 :	0	$4/9$	$4/9$	$4/9$	0	0	0	0
232 :	0	$1/3$	$1/3$	$1/3$	0	0	0	0
321 :	$2/3$	0	0	0	0	0	0	0
323 :	$1/3$	0	0	0	0	0	0	0

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