On Trasformations Between Belief Spaces

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Abstract: Commutative monoids of belief states have been defined by imposing one or more of the usual axioms and employing a combination rule. Familiar operations such as normalization and the Voorbraak map are surjective homomorphisms. The latter, in particular, takes values in a space of Bayesian states. The pignistic map is not a homomorphism between these same spaces. We demonstrate an impact this may have on robust decision making for frames of cardinality at least 3. We adapt the measure zero reflection property of some maps between probability spaces to define a category of belief states having plausibility zero reflecting functions as morphisms. Our definition encapsulates a generalization of the notion of absolute continuity to the context of belief spaces. We show that the Voorbraak map is a functor valued in this category.

1 Preliminaries

For a set U, $\mathcal{P}(U)$ is its powerset and $|U|$ is its cardinality. If $V \subset U$, then $U \backslash V$ is the complement of V in U. $\mathbb{R}$ is the set of real numbers and ϕ is the empty set. For a function $f : A \to B$ with source A and target B, $f^* : \mathcal{P}(B) \to \mathcal{P}(A)$ is its inverse image function and $f_* : \mathcal{P}(A) \to \mathcal{P}(B)$ is its direct image function. If $U \subset A$, for example, then $f_*(U) = \{b \in B \mid \exists a \in U.\ f(a) = b\}$. If f is a bijection then $f^{-1} : B \to A$ is its inverse. We make frequent, implicit use of the following corollary of the Binomial Theorem:

Lemma 1. *If X is a finite set and $U \subset W \subset X$, then*

$$\sum_{U \subset V \subset W} (-1)^{|V|} = \begin{cases} (-1)^{|W|} & \text{if } U = W; \\ 0 & \text{if } U \neq W. \end{cases}$$

Throughout this article, X will denote a finite, non-empty set to be called a *frame*.

2 Belief Representations

A function $m : \mathcal{P}(X) \to \mathbb{R}$ is a *belief state*. The set of all such is $\mathcal{X}$. For $U \subset X$, $\nu_U \in \mathcal{X}$ is defined by $\nu_U(V) = 1$ if $V = U$ and 0 otherwise. In particular, ν_X is *vacuous* and ν_ϕ is *conflicted*. The *zero* belief state assigns $0 \in \mathbb{R}$ to each $U \subset X$ and is denoted $0 \in \mathcal{X}$. $u \in \mathcal{M}$ defined by $u(U) = 1/|X|$ if $|U| = 1$ and 0 otherwise is *uniform*.

$m \in \mathcal{X}$ is *non-negative* if $0 \leq m(U)$ for all $U \subset X$. m is *unitary* if $1 = \sum_{U \subset X} m(U)$. m is *consonant* if $m(\phi) = 0$. Let $\mathcal{X}_+$, $\mathcal{X}_\Sigma$, and $\mathcal{X}_0$ respectively be the sets of non-negative, unitary, and consonant belief states. Define $\mathcal{X}_\mathcal{D} = \mathcal{X}_\Sigma \cap \mathcal{X}_+$, $\mathcal{X}_\mu = \mathcal{X}_\mathcal{D} \cap \mathcal{X}_0$, $\mathcal{X}_0' = \mathcal{X}_0 \cup \{\nu_\phi\}$, and $\mathcal{X}_\mu' = \mathcal{X}_\mu \cup \{\nu_\phi\}$. m is *Bayesian* if $m \in \mathcal{X}_\mu$ and $m(U) \neq 0$ implies $|U| = 1$. The uniform belief state, u, for example, is Bayesian. Let $\mathcal{X}_\mathcal{B}$ be the set of Bayesian belief states and $\mathcal{X}_\mathcal{B}' = \mathcal{X}_\mathcal{B} \cup \{0\}$.

Various alternative belief state representations are useful for proving theorems and gaining insights. Define $\beta : \mathcal{X} \to \mathcal{X}$ and $\beta^{-1} : \mathcal{X} \to \mathcal{X}$ by

$$\beta(m)(U) = \sum_{V \subset U} m(V) \qquad \text{and} \qquad \beta^{-1}(b)(U) = \sum_{V \subset U} (-1)^{|U \setminus V|}\, b(V)$$

$\beta(m)$ is the *implicability* of m. Proof that β and β^{-1} are inverse functions employs Lemma 1. Define $\kappa : \mathcal{X} \to \mathcal{X}$ and $\kappa^{-1} : \mathcal{X} \to \mathcal{X}$ by

$$\kappa(m)(U) = \sum_{U \subset V} m(V) \qquad \text{and} \qquad \kappa^{-1}(q)(U) = \sum_{U \subset V} (-1)^{|V \setminus U|}\, q(V)$$

$\kappa(m)$ is the *commonality* of m. Define $\lambda : \mathcal{X} \to \mathcal{X}$ and $\lambda^{-1} : \mathcal{X} \to \mathcal{X}$ by

$$\lambda(m)(U) = \begin{cases} \displaystyle\sum_{V \cap U \neq \phi} m(V) & \text{if } U \neq \phi; \\ 1 - \displaystyle\sum_{V \subset X} m(V) & \text{if } U = \phi; \end{cases}$$

$$\lambda^{-1}(\ell)(U) = \begin{cases} \displaystyle\sum_{V \subset U} (-1)^{|U \setminus V|}\, (1 - \ell(X \setminus V)) & \text{if } U \neq \phi \text{ and } U \neq X; \\ \ell(\phi) + \displaystyle\sum_{V \subset X} (-1)^{|X \setminus V|}\, (1 - \ell(X \setminus V)) & \text{if } U = X; \\ 1 - \ell(X) - \ell(\phi) & \text{if } U = \phi. \end{cases}$$

$\lambda(m)$ is the *plausibility* of m. For example, $\lambda(\nu_\phi) = 0$ and $\lambda^{-1}(0) = \nu_\phi$. Note that $\lambda(m)(\{x\}) = \kappa(m)(\{x\})$ for each $x \in X$. Define $\Lambda(m) = \sum_{x \in X} \lambda(m)(\{x\})$. The *Bayesian constant* of m is

$$\mathcal{B}(m) = \begin{cases} 1/\Lambda(m) & \text{if } \Lambda(m) \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Composites of the formulas above yield mappings between belief representations. Maps between implicability, b, and commonality, q, are:

$$(\kappa \circ \beta^{-1})(b)(U) = \sum_{V \subset U} (-1)^{|V|}\, b(X \setminus V) \qquad \text{and} \qquad (\beta \circ \kappa^{-1})(q)(U) = \sum_{V \subset X \setminus U} (-1)^{|V|}\, q(V).$$

Transformations between implicability, b, and plausibility, ℓ, are:

$$(\beta \circ \lambda^{-1})(\ell)(U) = \begin{cases} 1 - \ell(X \setminus U) - \ell(\phi) & \text{if } U \neq \phi; \\ 1 - \ell(\phi) & \text{otherwise} \end{cases}$$

$$(\lambda \circ \beta^{-1})(b)(U) = \begin{cases} b(X) - b(X \setminus U) & \text{if } U \neq X; \\ 1 - b(X) & \text{otherwise.} \end{cases}$$

Between commonality, q, and plausibility, ℓ, the following hold.

$$(\lambda \circ \kappa^{-1})(q)(U) = \begin{cases} \displaystyle\sum_{\phi \neq V \subset U} (-1)^{|V|+1}\, q(V) & \text{if } U \neq \phi; \\ 1 - q(\phi) & \text{otherwise} \end{cases}$$

$$(\kappa \circ \lambda^{-1})(\ell)(U) = \begin{cases} \displaystyle\sum_{\phi \neq V \subset U} (-1)^{|V|+1}\, \ell(V) & \text{if } U \neq \phi; \\ 1 - \ell(\phi) & \text{otherwise.} \end{cases}$$

The following are equivalent: m is consonant; $m(\phi) = 0$; $\beta(m)(\phi) = 0$; $\lambda(m)(X) = \beta(m)(X)$; $0 = \sum_{V \subset X} (-1)^{|V|}\, \kappa(m)(V)$.

3 Combination Operators

The *unnormalized combination* operator $\odot : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ defined by

$$(m \odot n)(U) = \sum_{A \cap B = U} m(A)\, n(B).$$

satisfies the following for all m, n, and $p \in \mathcal{X}$.

1. $m \odot n = n \odot m$
2. $m \odot \nu_X = m$
3. $(m \odot \nu_\phi)(U) = \left(\sum_{A \subset X} m(A) \right) \nu_\phi(U)$
4. $\kappa(m \odot n)(U) = \kappa(m)(U) \cdot \kappa(n)(U)$ for all $U \subset X$
5. $m \odot (n \odot p) = (m \odot n) \odot p$
6. $\lambda(m \odot n)(\{x\}) = \lambda(m)(\{x\}) \cdot \lambda(n)(\{x\})$ for all $x \in X$

Proofs of 1–3 are direct calculations. 4 follows from the fact that

$$\sum_{U \subset V} (m \odot n)(V) = \sum_{U \subset V} \sum_{A \cap B = V} m(A)\, n(B) = \sum_{U \subset A} \sum_{U \subset B} m(A)\, n(B) = \left(\sum_{U \subset A} m(A) \right) \left(\sum_{U \subset B} n(B) \right).$$

5 follows from 4 and the fact that κ and κ^{-1} are inverse functions:

$$(m \odot n) \odot p = \kappa^{-1}(\kappa((m \odot n) \odot p) = \kappa^{-1}(\kappa(m)\, \kappa(n)\, \kappa(p)) = \kappa^{-1}(\kappa(m \odot (n \odot p))).$$

6 follows from 4 and the fact that $\kappa(m)$ and $\lambda(m)$ agree on singletons.

The *combination* operator $\oplus : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is defined by

$$(m \oplus n)(U) = \begin{cases} \frac{(m \odot n)(U)}{1 - (m \odot n)(\phi)} & \text{if } (m \odot n)(\phi) \neq 1 \text{ and } U \neq \phi; \\ 0 & \text{if } (m \odot n)(\phi) \neq 1 \text{ and } U = \phi; \\ \nu_\phi(U) & \text{otherwise} \end{cases}$$

and satisfies the following for all m, n, and $p \in \mathcal{X}$.

1. $m \oplus n \in \mathcal{X}'_0$
2. $m \oplus n = n \oplus m$
3. $m \oplus \nu_X = m$ iff $m \in \mathcal{X}'_0$
4. $m \oplus \nu_\phi = \nu_\phi$ iff $m \in \mathcal{X}_\Sigma$
5. commonality
6. if m, $n \in \mathcal{X}_\mathcal{D}$, then $m \oplus n \in \mathcal{X}_\mathcal{D}$
7. if m, n, $p \in \mathcal{X}'_\mu$, then $m \oplus (n \oplus p) = (m \oplus n) \oplus p$
8. if m, $n \in \mathcal{X}'_\mathcal{B}$, then $m \oplus n \in \mathcal{X}'_\mathcal{B}$
9. if $m \in \mathcal{X}'_\mathcal{B}$, then $m \oplus u = m$

For $m \in \mathcal{X}$, define $\odot_0\, m = m$ and, for $n \geq 1$, $\odot_n\, m = m \odot (\odot_{n-1}\, m)$. Similar definitions apply to iteration of $\oplus$.

4 Belief State Monoids and Homomorphisms

Below we review the commutative monoids $(\mathcal{X}_+, \odot, \nu_X)$, $(\mathcal{X}_\mathcal{D}, \odot, \nu_X)$, and $(\mathcal{X}'_\mu, \oplus, \nu_X)$ and the homomorphisms Φ_Σ and Φ_0 introduced in [2] and the commutative monoid $(\mathcal{X}'_\mathcal{B}, \oplus, u)$ and the homomorphism $\mathfrak{V}$ implicit in [6].

Restriction of $\odot$ gives an operator $\mathcal{X}_+ \times \mathcal{X}_+ \to \mathcal{X}_+$. Properties 1, 2, and 5 of $\odot$ listed in Section 3 establish that $(\mathcal{X}_+,\, \odot,\, \nu_X)$ is a commutative monoid. Another restriction of $\odot$ produces an operator $\mathcal{X}_\Sigma \times \mathcal{X}_\Sigma \to \mathcal{X}_\Sigma$. Since $\mathcal{X}_\mathcal{D} = \mathcal{X}_+ \cap \mathcal{X}_\Sigma$, $(\mathcal{X}_\mathcal{D},\, \odot,\, \nu_X)$ is also a commutative monoid. $m \odot \nu_\phi = \nu_\phi$ for $m \in \mathcal{X}_\mathcal{D}$ by property 3 of $\odot$.

The *normalization operator* $\Phi_\Sigma : \mathcal{X}_+ \to \mathcal{X}_\mathcal{D}$ defined by

$$\Phi_\Sigma(m)(U) = \begin{cases} m(U) \Big/ \sum\limits_{V \subset X} m(V) & \text{if } m \neq 0 \\ \nu_\phi(U) & \text{otherwise} \end{cases}$$

is a monoid homomorphism. That is, $\Phi_\Sigma(m \odot n) = \Phi_\Sigma(m) \odot \Phi_\Sigma(n)$ and $\Phi_\Sigma(\nu_X) = \nu_X$. Moreover, if $m \in \mathcal{X}_\mathcal{D}$, then $\Phi_\Sigma(m) = m$.

Restriction of $\oplus$ gives an operator $\mathcal{X}'_\mu \times \mathcal{X}'_\mu \to \mathcal{X}'_\mu$ by properties 1 and 6 of $\oplus$ listed in Section 3. Properties 2, 3, and 7 establish that $(\mathcal{X}'_\mu, \oplus, \nu_X)$ is a commutative monoid. $m \oplus \nu_\phi = \nu_\phi$ for $m \in \mathcal{X}'_\mu$ by property 4. Another restriction of $\oplus$ gives an operator $\mathcal{X}'_\mathcal{B} \times \mathcal{X}'_\mathcal{B} \to \mathcal{X}'_\mathcal{B}$ by property 8. The results cited above together with property 9 establish that $(\mathcal{X}'_\mathcal{B}, \oplus, u)$ is a commutative monoid. Moreover, $m \oplus 0 = 0$ in $\mathcal{X}'_\mathcal{B}$.

The operator $\Phi_0 : \mathcal{X}_\mathcal{D} \to \mathcal{X}'_\mu$ defined by

$$\Phi_0(m)(U) = \begin{cases} m(U)/(1 - m(\phi)) & \text{if } m(\phi) \neq 1 \text{ and } U \neq \phi; \\ 0 & \text{if } m(\phi) \neq 1 \text{ and } U = \phi; \\ \nu_\phi(U) & \text{otherwise} \end{cases}$$

is a monoid homomorphism $(\mathcal{X}_\mathcal{D},\, \odot,\, \nu_X) \to (\mathcal{X}'_\mu, \oplus,\, \nu_X)$ and satisfies $\Phi_0(m) = m$ for $m \in \mathcal{X}'_\mu$.

The Voorbraak map $\mathfrak{V} : \mathcal{X}'_\mu \to \mathcal{X}'_\mathcal{B}$ is defined by

$$\mathfrak{V}(m)(U) = \begin{cases} \lambda(m)(\{x\})/\Lambda(m) & \text{if } \Lambda(m) \neq 0 \text{ and } U = \{x\}; \\ 0 & \text{otherwise} \end{cases}$$

gives a monoid homomorphism $(\mathcal{X}_\mu,\, \oplus,\, \nu_X) \to (\mathcal{X}_\mathcal{B},\, \oplus,\, u)$ satisfying $\mathfrak{V}(m) = m$ for $m \in \mathcal{X}'_\mathcal{B}$. [Note that if $m \in \mathcal{X}_\mathcal{D}$, then $\Lambda(m) = 0$ iff $m = \nu_\phi$. This permits a simplification of $\mathfrak{V}$.]

The diagrams below summarizes the results described in this section. The first illustrates the commutative monoids and monoid homomorphisms.

$$(\mathcal{X}_+, \odot, \nu_X) \xrightarrow{\Phi_\Sigma} (\mathcal{X}_\mathcal{D}, \odot, \nu_X) \xrightarrow{\Phi_0} (\mathcal{X}'_\mu, \oplus, \nu_X) \xrightarrow{\mathfrak{V}} (\mathcal{X}'_\mathcal{B}, \oplus, u)$$

The underlying functions of these homomorphims are the horizontal arrows of the commutative diagram below.

$$\begin{array}{ccccccc} \mathcal{X}_+ & \xrightarrow{\Phi_\Sigma} & \mathcal{X}_\mathcal{D} & \xrightarrow{\Phi_0} & \mathcal{X}'_\mu & \xrightarrow{\mathfrak{V}} & \mathcal{X}'_\mathcal{B} \\ & \nwarrow & \Uparrow & \nwarrow & \Uparrow & \nwarrow & \Uparrow \\ & & \mathcal{X}_\mathcal{D} & & \mathcal{X}'_\mu & & \mathcal{X}'_\mathcal{B} \end{array}$$

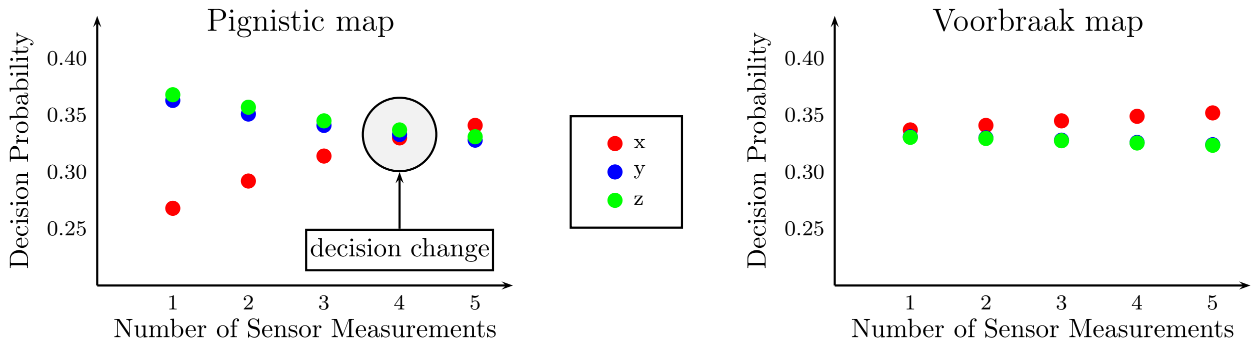
Vertical arrows are identity functions and diagonals are inclusions. Surjectivity of Φ_Σ, Φ_0, and $\mathfrak{V}$ are consequences of commutativity (in any category, $g \circ f$ an epimorphism implies g an epimorphism).

5 Robust Decision Making

Conversion of a belief state $m \in \mathcal{X}_\mu$ to a Bayesian state is an approach to decision making with belief models. In [1], Cobb and Shenoy observed that the pignistic map may display a decision change when applied to iterates $\oplus_k m$ of a fixed m. Recall that the pignistic map $\mathfrak{P} : \mathcal{X}_\mu \to \mathcal{X}_\mathcal{B}$ is defined by

$$\mathfrak{P}(m)(U) = \begin{cases} \sum_{x \in V} m(V)/|V| & \text{if } U = \{x\}; \\ 0 & \text{otherwise.} \end{cases}$$

The figures below illustrate this phenomenon for m defined on the frame $X = \{x, y, z\}$. The horizontal axes give the index k of $\oplus_k m$. The vertical axes in the left figure indicate values of $\mathfrak{P}(\oplus_k m)$ on singletons. In the right figure the vertical axis indicates values of $\mathfrak{V}(\oplus_k m)$ on singletons. The axis labels indicate an interpretation of the model in which one or more 'independent sensors' receive a fixed input signal m. Intuitively, reinforcement of the fixed signal should increase discrimination and decision confidence. The pignistic map, however, displays a decision change.

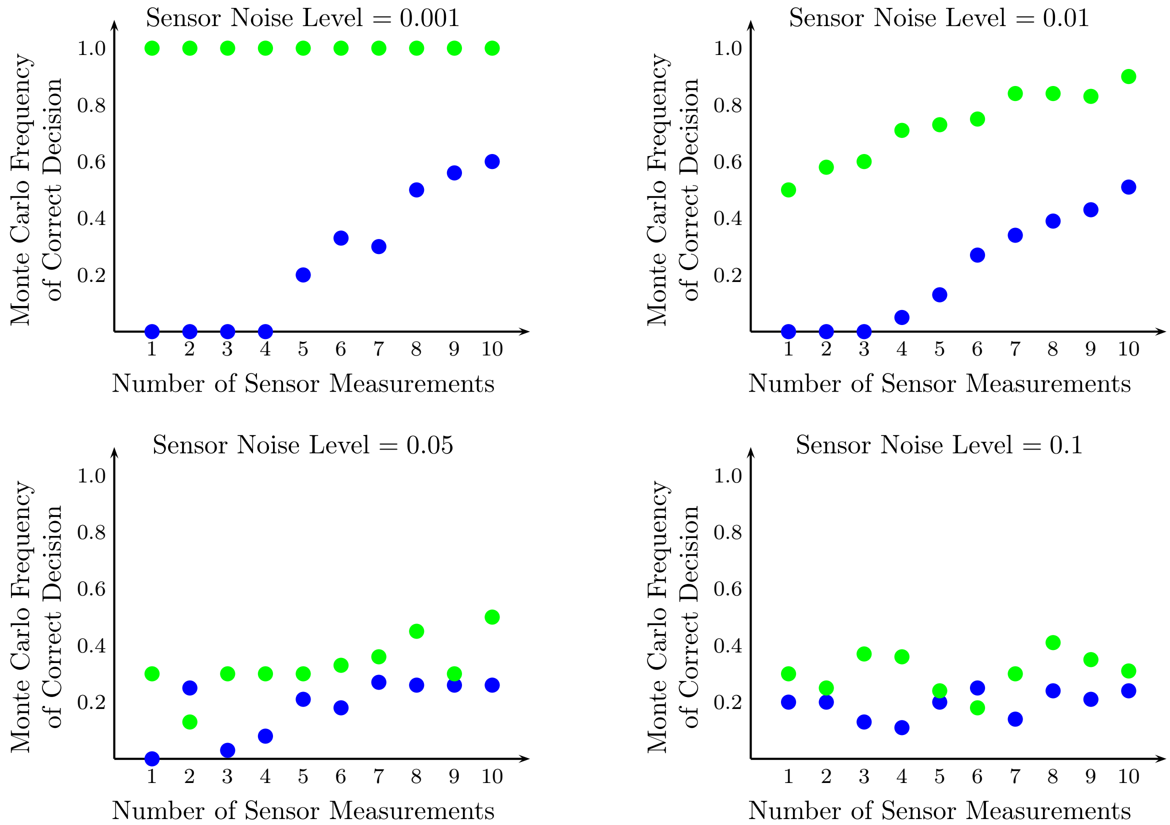


The above example is not an isolated one. It is computed using one member of the following two parameter family (with $\delta = 0.2$ and $\epsilon = 0.1$) defined for sufficiently small δ and ϵ by

U	ϕ	$\{x\}$	$\{y\}$	$\{z\}$	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$	X
$m(U)$	0	0	δ	$\delta + \epsilon$	$\delta + 2\,\epsilon$	$\delta + \epsilon$	0	$1 - 4\,(\delta + \epsilon)$

This family examples is not unique for the $|X| = 3$. Although no such decision change can be observed if $|X| = 2$, decision errors can be observed in families of belief states defined on any frame with $|X| \geq 3$ since the $|X| = 3$ case can be embedded in such higher-dimensional cases.

In real-world applications, belief states derived from sensor or human imputs are subject to random noise. Since $\odot$ is continuous (assuming the usual topology on $\mathbb{R}$ and identifying $\mathcal{X}$ with $\mathbb{R}^{|X|}$), we expect the pignistic map to display similar decision changes in noisy environments. The four plots in the next figure illustrate the effect of simulated sensor noise. The noise model simply perturbs non-zero belief state values by a gamma distribution and zero values by an exponential then renormalizes the resulting belief state. The 'noise level' indicates the magnitude of the variance of the perturbation random variables. In each case the Voorbraak map correctly assigns the highest probability to the correct singleton, $\{x\}$, with greater frequency than the pignistic map does in Monte Carlo simulations with a trial size of 20.



6 Categories of Belief States

If $\mathcal{C}$ is a (locally-small) category, then $|\mathcal{C}|$ is its class of objects and $\mathcal{C}(A, B)$ is the set of morphisms having source A and target B. Category-theoretic definitions used are from [4].

In [7], Wendt defined and investigated two categories having probability spaces as objects. The first has measure zero reflecting functions as morphisms while the latter has categorical disintegrations. These constructions led to the establishement of a semantics of intuitionistic higher-order predicate calculus employing probability spaces [3]. Such connections between rules-based systems and uncertainty models are of interest for their potential applications to information fusion systems and human/machine interfaces. Below we take first steps in adapting Wendt's constructions to belief models.

A *probability density* on a finite, non-empty set X is a function $p : X \to \mathbb{R}$ satisfying $0 \le p(x)$ and $1 = \sum_{x \in X} p(x)$. A *probability space* is a pair (X, p) with p probability density on X. Assume such spaces are equipped with powerset σ-algebras. Let $\widehat{p}$ be the induced measure defined by $\widehat{p}(U) = \sum_{x \in U} p(x)$. If (X, p) and (Y, q) are probability spaces, $f : X \to Y$ is *measure zero reflecting* if $\widehat{q}(V) = 0$ implies $\widehat{p}\,(f^*(V)) = 0$ (i.e., the image measure $\widehat{p} \circ f^*$ on Y is absolutely continuous with repect to $\widehat{q}$). Let **M0RP** be the category having probability spaces as objects and measure zero reflecting functions as morphisms. In **M0RP**, identity functions are identity morphisms and composition is ordinary function composition. The finiteness assumption and restriction to probability measures makes **M0RP** a proper subcategory of **M0R** defined in [7].

A *belief space* is a pair (X, m) with X finite, non-empty set and $m \in \mathcal{X}_\mu$. If (X, m) and (Y, n) are belief spaces, a function $f : X \to Y$ is *plausibility zero reflecting* if, given $V \subset Y$,

$$\lambda(n)(V) = 0 \;\text{ implies } \lambda(m)\left(f^*(V)\right) = 0.$$

Let **P0R** be the category having belief spaces as objects and plausibility zero reflecting functions as morphisms. Identity morphisms and composition in **P0R** are defined as in **M0RP**. Composition is well defined: if $f \in \mathbf{P0R}((X,m),\,(Y,n))$, $g \in \mathbf{P0R}((Y,n),\,(Z,r))$, and $W \subset Z$, then $\lambda(r)(W) = 0$ implies $\lambda(n)(g^*(W)) = 0$ implies $\lambda(m)(f^*(g^*(W))) = \lambda(m)((g \circ f)^*(W)) = 0$.

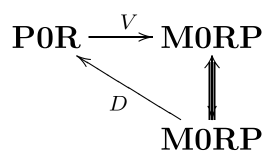
Theorem 1. *There is a faithful functor* $D : \mathbf{M0RP} \to \mathbf{P0R}$ *defined on objects by* $D(X,\,p) = (X,\,d(p))$ *where* $d(p)(\{x\}) = p(x)$ *and* $d(p)(U) = 0$ *for* $|U| \neq 1$. *On morphisms,* $D(f) = f$.

$D(X,\,p) \in |\mathbf{P0R}|$ since $d(p) \in \mathcal{X}_{\mathcal{B}} \subset \mathcal{X}_\mu$. Observe that

$$\lambda(d(p))(A) = \sum_{U \cap A \neq \phi} d(p)(U) = \sum_{x \in A} d(p)(\{x\}) = \sum_{x \in A} p(x) = \widehat{p}(A)$$

for $\phi \neq A \subset X$. If $f \in \mathbf{M0RP}((X,p),\,(Y,q))$, then $D(f)$ is plausibility zero reflecting since, $0 = \lambda(d(q))(B) = \widehat{q}(B)$ implies $0 = \widehat{p}(f^*(B)) = \lambda(d(p))(f^*(B))$.

Theorem 2. *The Voorbraak map induces a faithful functor* $V : \mathbf{P0R} \to \mathbf{M0RP}$ *defined on objects by* $V(X,\,m) = (X,\,v(m))$ *where* $v(m)(x) = \mathfrak{V}(m)(\{x\})$. *On morphisms* $V(f) = f$. *Moreover, the diagram below of categories and functors is commutative*



where the vertical arrow is the identity functor.

$V(X,\,m) \in |\mathbf{M0RP}|$ since $m \in \mathcal{X}_\mu$ implies $0 < \Lambda(m)$ which implies $\mathfrak{V}(m) \in \mathcal{X}_{\mathcal{B}}$. To verify that $V(f)$ is measure zero reflecting if $f \in \mathbf{P0R}((X,m),\,(Y,n))$, note that

$$0 = \widehat{v(n)}(B) = \sum_{y \in B} v(n)(y) = \sum_{y \in U} \lambda(n)(\{y\})/\Lambda(n)$$

implies $0 = \lambda(n)(\{y\})$ for each $y \in B$. This implies $0 = \lambda(m)(f^*(\{y\}))$ for each $y \in B$, hence,

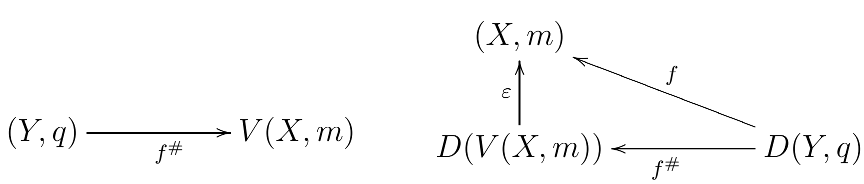
$$0 = \widehat{v(m)}(f^*(\{y\})).$$

Disjointness of the sets $f^*(\{y\})$ for distinct y and additivity of $\widehat{v(m)}$ yields $0 = \widehat{v(m)}(f^*(U))$.

Commutativity of the diagram implies that D is injective on objects, hence, **M0RP** can be identified as a subcategory of **P0R** with D as embedding.

Theorem 3. *: The Voorbraak functor* $V : \mathbf{P0R} \to \mathbf{M0RP}$ *is right adjoint to the functor* $D : \mathbf{M0RP} \to \mathbf{P0R}$.

For a belief space (X,m) we must show that the function $\varepsilon : X \to X$ defined by $\varepsilon(x) = x$ is a plausibility zero reflecting map $\varepsilon : D(V(X,m)) \to (X,m)$ and that, given any probability space (Y,q) and plausibility zero reflecting map $f : D(Y,q) \to (X,m)$, there is a unique $f^{\#} \in \mathbf{M0RP}((Y,q),\,V(X,m))$ for which the diagram below right is commutative.



M0RP **P0R**

To establish the condition on ε, assume $\lambda(m)(U) = 0$. Note that

$$\lambda(d(v(m)))(U) = \widehat{v(m)}(U) = \sum_{x \in U} \lambda(m)(\{x\})/\Lambda(m).$$

If $x \in U$ and $0 < \lambda(m)(\{x\})$, then there exists $A \subset X$ with $x \in A$ and $0 < m(A)$. $x \in A$ implies $A \cap U \neq \phi$, hence $0 < m(A) \leq \lambda(m)(U)$ contradicting the assumption on U. This implies $\lambda(d(v(m)))(U) = 0$, hence, ε is plausibility zero reflecting.

We expect that Dempster's Combination Rule induces a symmetric monoidal structure on **P0R** just as product measures induce such a structure on **M0RP**. The tensor product on **P0R** is defined as follows. For belief spaces, (X, m) and (Y, n), let $X \xleftarrow{\pi} X \times Y \xrightarrow{\pi'} Y$ be the cartesian product and associated projection functions. Define $(X, m) \otimes (Y, n) = (X \times Y, m \otimes n)$ where

$$m \otimes n = \Phi_0(\Phi_\Sigma(m \circ \pi_*)) \oplus \Phi_0(\Phi_\Sigma(n \circ \pi'_*)).$$

Development of this monoidal structure will be the topic of a later article.

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