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ON CATEGORIES OF COHESIVE, ACTIVE SETS AND OTHER DYNAMIC SYSTEMS

BY

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THESIS

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Abstract

This work is intended to contribute to a program of developing general concepts and methods for applying category theory to the modeling and solution of scientific problems. It was motivated by my experiences as an engineering student studying continuum and kinetic models of fluid flows. The language of category theory is employed because it facilitates precise comparisons between diverse types of structures. Using this language to investigate relationships between fluid flow models requires categorical specifications of constitutive relations and of idioms occurring across classifications of dynamic systems.

It is shown that a category-theoretic definition of chaotic system applies not only to the Smale horseshoe, a standard chaotic system, but also to Conway's "Game of Life" automaton. Symbolic dynamics of the "Dining Philosophers" relational system is computed. A category composed of stochastic matrices is defined and some of its elementary properties are developed. A categorical variant of symbolic dynamics is applied to a finite stochastic process. Using point-wise Kan extension formulas, conditions ensuring existence of certain representations among categories of dynamic systems are proved.

For my parents

Acknowledgments

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Guide to Notation

Naming of a category after its objects is indicated by the phrase: ‘category of ...’ (e.g. the category of sets) while naming after its morphisms is indicated by: ‘category composed of ...’ (e.g. the category composed of stochastic matrices).

A diagram such as $\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ & \swarrow \text{---} & \\ C & \xrightarrow{\quad} & D \end{array}$ is described as in a category \mathcal{C} if the nodes are labels for objects of \mathcal{C} and the arrows are labels for morphisms. It is also an abbreviation for certain source-target information. A diagram in a category is said to commute if all composite paths between fixed nodes are equal. All diagrams are assumed

to commute unless indicated otherwise. $\begin{array}{ccc} A & \xrightarrow{u} & B \\ x \downarrow & \swarrow y & \downarrow v \\ C & \xrightarrow{w} & D \end{array}$ means

$$v \circ u = w \circ x, \quad w \circ y = v, \quad \text{and} \quad y \circ u = x$$

in addition to the source-target information. A puncture mark $+$ removes one equation. $\begin{array}{ccc} A & \xrightarrow{u} & B \\ x \downarrow & \swarrow y & \downarrow v \\ C & \xrightarrow{w} & D \end{array} +$ means

$$v \circ u = w \circ x \quad \text{and} \quad y \circ u = x$$

while $\begin{array}{ccc} X & \xrightarrow{u} & B \\ x \downarrow & \swarrow y & \downarrow v \\ C & \xrightarrow{w} & D \end{array} +$ still implies $v \circ u = w \circ x$. Commutativity of $\begin{array}{c} \text{f} \\ \circlearrowleft \\ A \\ \text{id} \end{array}$ means $f = id_A$. A puncture

does not assert the negation of an equation. $A \begin{array}{c} \text{f} \\ \circlearrowleft \\ + \\ \text{g} \end{array} B \xrightarrow{m} C$ means $m \circ f = m \circ g$ but asserts neither $f = g$ nor $f \neq g$.

\mathbf{N} denotes the set of natural numbers $\{0, 1, \dots\}$, \mathbf{Z} is the set of integers, and \mathbf{R}^+ is the set of nonnegative reals. $n \in \mathbf{N}$ is the set $\{0, 1, \dots, n - 1\}$.

1. Introduction

Every concept arises from the equation of unequal things. Just as it is certain that one leaf is never totally the same as another, so it is certain that the concept “leaf” is formed by arbitrarily discarding these individual differences and by forgetting the distinguishing aspects. . . . What then is truth? A movable host of metaphors, metonymies, and; anthropomorphisms: in short, a sum of human relations which have been poetically and rhetorically intensified, transferred, and embellished, and which, after long usage, seem to a people to be fixed, canonical, and binding. Truths are illusions which we have forgotten are illusions. . . . it is originally **language** which works on the construction of concepts, a labor taken over in later ages by **science**.

Friederich Nietzsche

“On Truth and Lies in a Nonmoral Sense” (1873)

A theory is a mathematical model for an aspect of nature. One good theory extracts and exaggerates some facets of truth. Another good theory may idealize other facets. A theory cannot duplicate nature, for if it did so in all respects, it would be isomorphic to nature itself and hence useless, a mere repetition of all the complexity which nature presents to us, that very complexity we frame theories to penetrate and set aside. . . . With this sober and critical understanding of what a theory is, we need not see any philosophical conflict between two theories, one of which represents a gas as a plenum, the other as a numerous assembly of punctual masses. Models of either kind represent aspects of real gases; if they represent those properly, they should entail many of the same conclusions, though of course not all.

Clifford A. Truesdell and Robert G. Muncaster

Fundamentals of Maxwell’s Kinetic Theory of a Simple Monatomic Gas. (1980)

. . . in mathematical practice we must, more than in any other science, hold a given object quite precisely in order to construct, calculate, and deduce; yet we must also constantly transform it into other objects.

F. William Lawvere

“Some thoughts on the Future of Category Theory”
(1990)

1.1. On applied mathematics

We humans are amazing mathematicians. The fluent use of language is a mundane example of our rapid and graceful manipulation of models and representations of concepts. The social and political structures through which we interact and the rules of games we play are further illustrations. Each day we refine these structures, build new ones, and use them to make predictions about the world.

Applied and pure mathematicians are not so divorced in their professional activities. Both construct mathematical representations and investigate their properties. They are distinguished primarily by their motives for constructing models and by the kinds of problems on which they work. Members of the first group may study physical situations, engineering problems, or phenomena from other sciences. The latter study questions arising within mathematics and less directly connected to everyday experiences.

1.2. On continuum mechanics

This work was motivated by my experiences as an engineering student studying fluid dynamics and Maxwell's kinetic theory of gases. Sundry mathematical pieces share the quality of being applied to or motivated by modeling of fluid dynamics. Knot theory is used to study orbits of the Lorenz equations, introduced in 1963 to model atmospheric dynamics [BW]. Leif Arkeryd applied nonstandard analysis to prove existence of classes of solutions of the Maxwell-Boltzmann equation [TM]. In the twentieth century, the asymptotic series defined by Poincaré in 1886 and incorporated into models of boundary layer flow by Ludwig Prandtl in 1905 earned high status as tools of theoretical fluid mechanics [Ga], [Dy]. For his study of tides completed in 1840, Herman Günther Grassmann developed the first system of spatial analysis based on vectors [Cw]. At the origin of the vast literature of stochastic processes are observations made by botanist Robert Brown in 1827 of an irregular motion of particles suspended in water [Du]. This variety of ideas is a consequence of the diversity of fluid behavior: the material ventilating our offices and exchanging CO₂ for O₂ in the lawn outside is the same material which ablates exhausted satellites as they fall from orbit.

1.21. Observable qualities of fluid flows (such as density, velocity, and temperature) are gross manifestations of molecular states and activities [Bat2], [CC], [TM]. Viscous transports of mass, momentum, and energy are communications via molecular collisions and through translations occurring between encounters. In macroscopic theory, these transports are proportional to spatial rates of change of properties[†]: energy flux, for example, is proportional to the gradient of the temperature field. Diffusion, viscosity, and heat conduction coefficients are the proportionality constants.

In the kinetic theory of gasses, all properties are computed from the molecular density $F(t, x, v)$, introduced by James Clerk Maxwell in 1867 and assigning probabilities to molecular velocities at each time t and position x . A collisions operator \mathbf{C} , incorporating dynamic and statistical ideas, models the effects of molecular encounters on the evolution of F .

Physicists heuristically describe the kinetic theory as applicable when gross properties vary significantly over distances on the order of the mean-free-path (average distance molecules travel between collisions) and engineers describe transitions between flight regimes of high-altitude vehicles. In 1952, during the “golden age of hypersonic aerodynamics” [A], Truesdell derived a precise condition in this spirit [pages 57–60 of TM].

[†] **Newton described linearly viscous fluids in his 1687 *Principia* [Wh].**

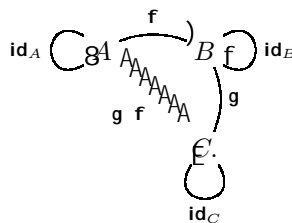
1.22. Relating these disparate mathematical models of the behavior of a single material has challenged mathematicians since Maxwell published his (second) kinetic theory. Early in the twentieth century, David Hilbert, Sydney Chapman, and David Enskog all developed methods for approximating solutions of the equation describing evolutions of molecular densities [CC], [TM]. In 1956, Ikenberry and Truesdell invented a method of approximating viscous transports in a gas of Maxwellian molecules[†]. The only known exact solutions of the Maxwell-Boltzmann equation are that of Max Krook and Tai Tsun Wu [KW] and those of Robert G. Muncaster [pages 284–291 of TM].

In his 1975 dissertation, Muncaster developed a direct method for approximating transport coefficients [Chapter XXIV of TM]. With insight gained from his investigation of specific examples, he clarified a general situation in which one model, the fine theory, may give detailed information about a dynamical system whereas a second, the coarse theory, yields cruder, more accessible results. He viewed a coarse theory as a special invariant subsystem and described an algorithm for deriving coarse theories from fine theories [Munc1]. Distillation of the Stokes-Kirchoff theory of ideal gases from Maxwell’s kinetic theory has this character. Muncaster characterized the status of the continuum theory as: aggregate, invariant, relaxed, and local [Munc2], [TM]. In 1986, F. W. Lawvere gave a category-theoretic discussion of the first two of these qualities [pages 5–6 of L5].

My long-term research program involves using the language and tools of category theory to investigate the relationship between models of fluids flows and to develop examples of dynamic systems sharing similar relationships. This work should incorporate Muncaster’s characterization as well as Walter Noll’s general theory of materials [Tr1], [1.7]. Paul Taylor wrote in the preface to his book on computing science [T] that: “[category theory] seems to me to be the most efficient heuristic tool for investigating structure, and comparing it in different examples.”

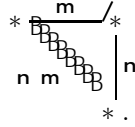
1.3. The language of category theory

A category is a mathematical structure consisting of objects and morphisms and equipped with operations which: specify domain and codomain objects for each morphism, assign an identity morphism to each object, compose two morphisms when the domain of one is the codomain of the other. Composition is associative and identity morphisms behave as such when composed. A finite category may be depicted by a directed graph:



[†] The intermolecular force between two Maxwellian molecules is proportional to the reciprocal of the f th power of the distance between them.

A monoid[†] $\mathcal{M} = (M, *, e)$ is a category with one object (denoted $*$), members of M as morphisms, the identity element (denoted e) as identity morphism, and composition given by the binary operation:

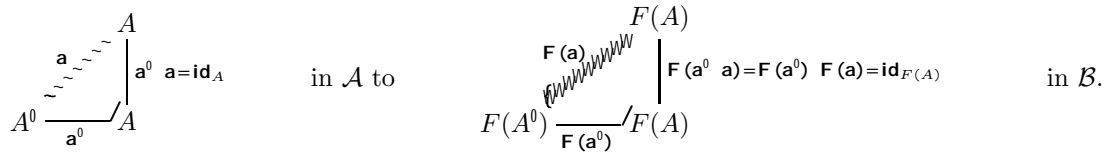


Groups are monoids with all morphisms invertible. A partially ordered set [II.5] $\mathcal{P} = (P, \leq)$ is a category having members of P as objects and a single morphism $p \rightarrow q$ whenever $p \leq q$. Examples of **large** categories include:

- Set: having sets as objects and functions as morphisms;
- Top: having topological spaces as objects and continuous maps as morphisms;
- Vect: having real vector spaces as objects and linear transformations as morphisms.

These may be construed as universes of mathematical discourse while monoids, posets, and other **small** categories typically act on objects of these universes.

1.31. Functors are the mappings between categories, assigning objects to objects and morphisms to morphisms and respecting the domain, codomain, identity, and composition operations. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ from a category \mathcal{A} to a category \mathcal{B} maps, for example, a diagram

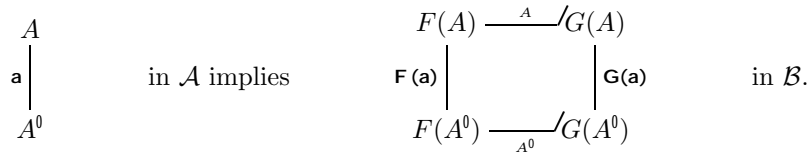


Functors $\text{Top} \rightarrow \text{Set}$ and $\text{Vect} \rightarrow \text{Set}$ arise from ignoring the extra structures in the domain categories. A functor $F : \mathcal{M} \rightarrow \text{Set}$, with \mathcal{M} a monoid, is an action of \mathcal{M} on a set: $F(*)$ is the set acted upon, each $F(m)$ is a transition $F(*) \rightarrow F(*)$. For a poset $\mathcal{P} = (P, \leq)$, a functor $F : \mathcal{P} \rightarrow \text{Set}$ assigns a set of states $F(p)$ to each $p \in P$ and specifies a transition rule $F(p) \rightarrow F(q)$ whenever $p \leq q$.

1.32. Natural transformations compare images of functors. Given $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{A} \rightarrow \mathcal{B}$, a natural transformation τ from F to G selects a \mathcal{B} -morphism

$$F(A) \xrightarrow{A} G(A)$$

for each \mathcal{A} -object in a coherent way:



[†] A monoid is a set equipped with an associative binary operation for which there is an identity element. See [II.3].

A natural transformation $\tau : F \Rightarrow G$ between monoid actions $\mathcal{M} \rightarrow \text{Set}$ is an equivariant map: $m \in M$ implies

$$\begin{array}{ccc} F(*) & \xrightarrow{\quad} & G(*) \\ \mathbf{F}(m) \Big\downarrow & & \Big\downarrow \mathbf{G}(m) \\ F(*) & \xrightarrow{\quad} & G(*) \end{array}$$

For a natural transformation $\tau : F \Rightarrow G$ between poset actions $\mathcal{P} \rightarrow \text{Set}$,

$$\begin{array}{ccc} F(p) & \xrightarrow{p} & G(p) \\ \mathbf{F}(p; q) \Big\downarrow & & \Big\downarrow \mathbf{G}(p; q) \\ F(q) & \xrightarrow{q} & G(q) \end{array}$$

whenever $p \leq q$. Appendix I includes a rigorous discussion of category theory.

1.4. A brief history of category theory

Categories, functors, and natural transformations were defined in 1945 by Samuel Eilenberg and Saunders Mac Lane to relate constructions in algebra and topology. During the following two decades category theory was vigorously developed by Charles Ehresmann and his students in France and in the United States by Eilenberg and Mac Lane, Michael Barr, Peter Freyd, John W. Gray, Gregory Maxwell Kelly, Daniel Kan, F. William Lawvere, Fred Linton, and Myles Tierney [MM]. A decisive step came in 1958 with Kan's definition of adjoints [K]. Freyd and Lawvere developed this central theme of category theory in the following years. Enriched categories were introduced by Jean Bénabou in 1963 [B1], [B2] and developed by Eilenberg and Kelly [EK], Mac Lane [Ma1], and Gray [Gr2].

Early applications of category theory were in pure mathematics. Eilenberg and Steenrod formulated axioms for homology theories and described their work in an influential 1952 text [ES]. Alexandre Grothendieck[†] used the language of category theory to generalize the notion of sheaf (a topological concept) in the early 1960s. His work led to rapid developments in algebraic geometry and to Pierre Deligne's 1974 solution of the Weyl conjectures for which he was awarded the Fields Medal in 1978. In 1965 Lawvere proposed a categorical[‡] foundation for mathematics, with properties specified abstractly rather than in terms of membership, and he gave an axiomatization for the category of sets [L2]. Five years later Lawvere and Tierney generalized Grothendieck's work with their notion of an elementary topos. Topos theory has been used to formulate smooth infinitesimal analysis, an approach to manifold theory and differential geometry [Be], [K], [MR]. It has also revealed connections between category theory and logic [Go], [J], [MM], [Mc].

In recent decades, category theory has been applied to more concrete problems. It provides a framework for studying the semantics of computer programming languages and concepts such as polymorphism. Robert

[†] See [C] for biographical information and a perspective on the influence of Grothendieck's research.

[‡] In this thesis, "categorical" is used in this sense: involving, according with, or considered with respect to specific categories. It will not mean: absolute. Robert Goldblatt [Go] invented "categorical" to have the former meaning.

Frank Carlslaw Walters used category theory to give abstract specifications of data types (stacks, queues, etc.) and an algebra for composing imperative programs. He and his colleagues at the University of Sydney and Universita dell' Insubria, Como, Italy have used bicategories to develop an algebra of asynchronous, distributed systems [KSW1], [KSW2], [We]. Concurrency modeling is an active research subject. Vaugh Pratt, Gordon Plotkin and R. J. van Glabbeek at Stanford base their model [Gu] on Chu spaces, named for and developed by P. H. Chu who was a master's student of Michael Barr at McGill University [Ba].

In support of these computing science researches, several groups are working to develop software for calculating with categories. Bob Rosebrugh at Mt. Allison University in Canada, Ronnie Brown in Bangor, Wales, and Anne Heyworth at Leicester, England are among those involved in this project [COMPCAT]. This work has exciting applications[†] to modeling computer networks and social systems [Dua]. Indeed, in 1944 John von Neumann and Oskar Morganstern wrote [NM]:

“It is to be expected — or feared — that mathematical discoveries of a stature comparable to that of calculus will be needed in order to produce decisive success in [mathematical economics].”

In the 1990s, John Baez and James Dolan became interested in using higher-dimensional categories as a formalism for constructing quantum field theories. Currently there are lively interactions between theoretical physicists and category theorists seeking the best definition of higher-dimensional categories [Bae], [Ka], [C].

Three themes may be distilled from studying successful applications of category theory. It permits the mathematical modeler, whose interests may be more or less abstract or concrete, to simultaneously discuss aspects of and interactions among different mathematical structures. Category theory is largely constructive: a categorical model provides algorithms and an algebra of composition for investigating properties of models. The perspective of category theory may suggest analogies among models.

1.5. Modeling with categories

In drawing parallels between thermodynamics and classical mechanics, Clifford Truesdell wrote [Tr2]:

“Any branch of mathematical physics is constructed in terms of

1. A list of **primitive quantities**, not defined except by mathematical properties laid down for them.
2. **De nitions** of other quantities in terms of the primitives.
3. **General axioms** stated as mathematical relations satisfied by the primitives and the defined quantities.
4. Proved **theorems** referring to
 - a. The theory as a whole, or
 - b. Mathematically defined special cases.”

[†] [Tr2] page 149: \That heat could sometimes cause mechanical e ect, and much of it, had been known since the disaster that befel STREPSIADES while he was cooking the haggis for the feast of Zeus, but apparently it was the sooty proliferation of the steam engine in the early nineteenth century that rst roused physicists to pay much attention to the phenomenon." Perhaps the world-wide-web as a vehicle for global commerce will stir-up interest in category theory. See [LOOP].

His words reveal a relationship between **mathematical modeling**, an activity of applied mathematicians, and **model theory**, a part of pure mathematics: mathematical logic may be of interest to applied mathematicians because it clarifies the distinction between manipulations of symbols (syntax) and interpretations of symbols (semantics) [Cr], [CK], [Go], [J].

Category theory bridges these uses of the root **model**. The theory of sketches was introduced by C. Ehresmann in 1968 and developed by René Guitart and Christian Lair [GL1], [GL2]. It is the starting-point of **categorical model theory**, a term invented in 1989 by Michael Makkai and Robert Paré [MP]. A **sketch** is a category (or graph [BW2]) equipped with prescribed features and used to present a kind of mathematical structure. A **model** of a sketch is a functor from the category and which satisfies conditions involving the extra features. Models in a fixed category \mathcal{C} form a new category. In 1981, Lair characterized those categories which arise as Set-valued models of some sketch [L], [BW1]. Gray showed that sketches themselves form a category with many desirable properties [Gr3]. His [Gr1] may aid researches involving categories of non-Set-valued models of sketches.

Theoretical computing science (informatics [T]) gives first examples of the utility of sketches in applied mathematics [BW2], [KSW], [W].

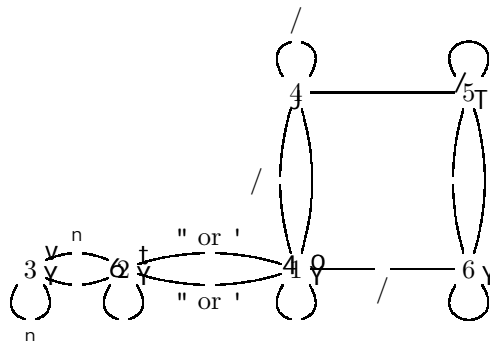
1.51. The design (syntax) of a computer program may be represented by a graph and its implementation (semantics) by a graph homomorphism to a category. Equivalently, semantics is a functor defined on the free category of the graph. Here is an example. The UNIX operating system (introduced in 1978 by Ken Thompson and Dennis Ritchie) and most of its applications programs are written in the C programming language [KR]. As an exercise in learning the language I wrote a program which reads a C source-code file and rewrites it with comments removed. C comments begin with `/*` and end with `*/` as in:

```

/* Copy input to output.  Program from [KR].  */
main() /* C programs begin execution with main.  */
{
    int c;
    while (( c = getchar()) != EOF) /* If there is an unread character, read it ... */
        putchar(c); /* ... then print it.  */
}

```

Syntax of my comment removal program is illustrated by



in which nodes indicate computation modes and arrows represent actions. The process is to begin in state $\text{READY} = 1$. Sequentially-read source-code characters induce transitions between modes. If a single or double quote is read while in mode 1, for example, the process transitions to state $\text{IN_QUOTE} = 2$. The decorated arrow $2 \xrightarrow{\text{Q}} 1$ indicates that the current character, a quote, should be output. $m \xrightarrow{\text{Q}} n$ indicates that the current character should be stored temporarily. $m \xrightarrow{\text{Q}} n$ means output both the current and the previous characters and $m \xrightarrow{\text{Q}} n$ means output a blank space. The character inducing a transition appears adjacent to the arrow. An unlabeled arrow from a node is a default, induced by all characters not already assigned to transitions from that node. When in mode $\text{IN_COMMENT} = 5$, for example, any character other than $*$ induces a transition back to mode 5 with no output or storage. There are sticky points to the design:

$b = b /* */ * a;$

is legal C code which should be read as

$b = b / * a;$

where $*a$ is a pointer.

Implementation is a graph homomorphism to the category of sets: to each node is assigned a set; to each arrow a function. Equivalently, it is a functor from the free category of the graph to the category of sets. Let \mathcal{A} denote the set of finite strings of ASCII characters. The empty string qualifies as finite. Let \mathcal{A}^0 be the set of strings having length at most one. Each node of the comment removal program may be interpreted as the set

$$\mathcal{A} \times \mathcal{A} \times \mathcal{A}^0.$$

The first entry gives the unread portion of the input file, the next is the written part of the output file, the third entry is for a possible stored character as in $1 \xrightarrow{\text{Q}} 4$. Each arrow is a function

$$\mathcal{A} \times \mathcal{A} \times \mathcal{A}^0 \xrightarrow{\text{Q}} \mathcal{A} \times \mathcal{A} \times \mathcal{A}^0.$$

A C program implementation appears in Appendix V. Real operating systems restrict sizes of files. In [W] Walters asks readers to construct representations of many finitely-presented categories. Michael Barr and Charles Wells [BW2 page 75] explained that:

“Probably the commonest type of research article that applies category theory to computing science is one that proposes some specific category other than Set to be the semantics of a certain kind of program or programming language.”

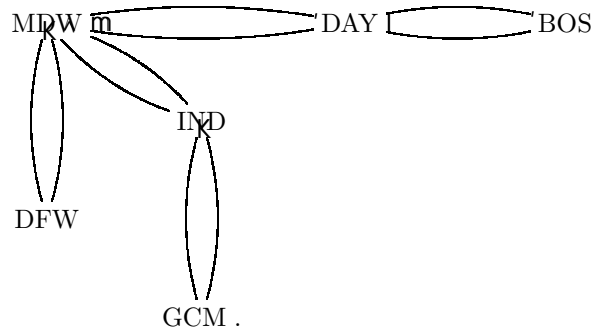
1.52. A map of airline service destinations is a model of the graph $\bullet \xrightarrow{\text{d}} \bullet$ in the category having sets as objects and binary relations as morphisms. The node may be interpreted, for example, as the set

$$X = \{\text{MDW}, \text{DFW}, \text{IND}, \text{GCM}, \text{DAY}, \text{BOS}\}$$

of airports serviced by the airline. L is the relation $\mathcal{L} : X \rightarrow X$ pronounced “links to,” tabulated, for example, by

$$A = \begin{array}{cc} \begin{array}{l} \circ \\ \swarrow \\ \circ \\ \swarrow \\ \circ \\ \swarrow \\ \circ \end{array} & \begin{array}{l} (\text{MDW, DFW}) \\ (\text{IND, MDW}) \\ (\text{IND, GCM}) \\ (\text{MDW, DAY}) \\ (\text{DAY, BOS}) \end{array} & \begin{array}{l} (\text{DFW, MDW}) \\ (\text{MDW, IND}) \\ (\text{GCM, IND}) \\ (\text{DAY, MDW}) \\ (\text{BOS, DAY}) \end{array} & \begin{array}{l} \circ \\ \swarrow \\ \circ \\ \swarrow \\ \circ \end{array} \end{array}$$

as depicted in



The composite $\mathcal{L} \circ \mathcal{L} : X \rightarrow X$ is the relation pronounced as “links with one connection to.”

1.53. In [L7] Lawvere explained that the graph

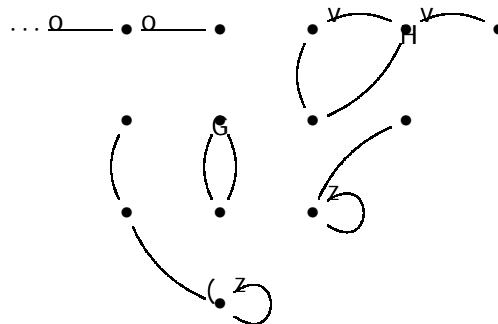


provides a syntax for genealogy. A model in the category of sets is a triple (X, m, f) with X a set, interpreted as a lineage or society. m and f are functions $X \rightarrow X$ respectively identifying the mother and father of each individual. He lists and interprets properties of categories of models and teaches how to add facets, such as gender, to models.

1.54. Robert Rosen [R1], [R2], [R3] was perhaps the first to suggest that category theory could be used to model physical phenomena. His papers influenced Andrée Charles Ehresmann and J.-P. Vanbremeersch [EV]. Missing from these are discussions of models.

1.55. Dynamic systems are structures changing over time and may be classified according to: the manner in which time advances, the nature of the state spaces and transition rules. Here are several examples. They are inducements for the slogan in [1.549].

1.551. The graph



illustrates a finite dynamic system $X : (\mathbf{N}, 0+) \rightarrow \text{Set}$: $X(*)$ is the set of dots, the arrows display $X(1) : X(*) \rightarrow X(*)$. To be a functor, $X : (\mathbf{N}, 0+) \rightarrow \text{Set}$ must preserve composites hence, $X(n) = X(1)^n$ [1.31].

1.552. A divisor of an integer i is an integer d for which $i = dq$ for some integer q . That is, d divides i . Euclid's algorithm computes the largest, common divisor of two integers, m and n . It is a functor $E : (\mathbf{N}, +) \rightarrow \text{Set}$. With $5 = \{0, 1, 2, 3, 4\}$, $E(*) = \mathbf{N}^3 \times 5$. $E(1) : E(*) \rightarrow E(*)$ is

$$E(1)(x, y, z, i) = \begin{matrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{matrix} \begin{matrix} (x, y, 0, 1) \\ (x, y, \text{remainder of } x \text{ divided by } y, 2) \\ (x, y, z, 3) \\ (0, y, 0, 4) \\ (y, z, z, 1) \\ (0, y, 0, 4) \end{matrix} \begin{matrix} \text{if } i = 0; \\ \text{if } i = 1; \\ \text{if } i = 2 \text{ and } z \neq 0; \\ \text{if } i = 2 \text{ and } z = 0; \\ \text{if } i = 3; \\ \text{if } i = 4. \end{matrix}$$

To determine the largest, common divisor d of m and n , compute the orbit of $(m, n, z, 0)$ for any $z \in \mathbf{N}$. The orbit reaches a fixed-point $(0, d, 0, 4)$ after a finite sequence of steps. The greatest common divisor of 18 and 12, for example, is 6:

$$\begin{array}{cccc} 18 & 12 & 0 & 0 \\ 18 & 12 & 0 & 1 \\ 18 & 12 & 6 & 2 \\ 18 & 12 & 6 & 3 \\ 12 & 6 & 6 & 1 \\ 12 & 6 & 0 & 2 \\ 0 & 6 & 0 & 4. \end{array}$$

The last state of the computation is a fixed-point of $E(1)$. Appendix V lists a Java program implementing this algorithm.

1.553. The flow determined by

$$\begin{pmatrix} x^0(t) \\ y^0(t) \end{pmatrix} = \begin{pmatrix} 0 & \pi \\ -2\pi & -2\pi \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

is a functor $Y : (\mathbf{R}^+, 0, +) \rightarrow \text{Set}$ with $Y(*) = \mathbf{R}^2$ and $Y(t) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} x(t, x_0, y_0) \\ y(t, x_0, y_0) \end{pmatrix} = \begin{pmatrix} x_0 \cos(\pi t) + (x_0 + y_0) \sin(\pi t) \\ y_0 \cos(\pi t) - (2x_0 + y_0) \sin(\pi t) \end{pmatrix} e^{-t}.$$

1.554. In the kinetic theory of gases, the molecular density F gives detailed information about the gas state: $F(t, x, \cdot)$ is a probability measure on molecular velocities at position x and time t . Truesdell and Muncaster display the basic axiom of kinetic theory as (equation III.38 of [TM])

$$F(t, x, v) = \mathcal{R}_{\mathbf{t}} \int_{\mathbf{t}_0}^{\mathbf{Z}_{\mathbf{t}}} F(t, x, v) + \mathcal{R}_{\mathbf{t}} \int_{\mathbf{s}} (\mathbf{C}F)(t, x, v) ds$$

having the following interpretation. A molecule having velocity v at place x and time t may occur for two reasons: it had velocity v at some other location and earlier time then moved to x without hitting other

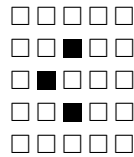
molecules; it had some other velocity but obtained v through collision then moved to x . The equation gives a functor $\mathcal{F} : (\mathbf{R}^+, \leq) \rightarrow \text{Set}$ with $\mathcal{F}(t)$ the set of admissible molecular densities and $F(s, t) : F(s) \rightarrow F(t)$ the transition rule [1.31].

1.555. Autonomic Computing is a research program recently proposed by I.B.M. to address the growing complexity of information technology systems[†]. It proposes the development of higher-level automation of computing using systems which respond and adapt to changes in their digital environments without human intervention, just as our autonomic nervous systems respond to changes in our physical environments.

John H. Conway's game "Life" is a simple model of birth, death, and emergence in living systems. Imagine points of \mathbf{Z}^2 to be **cells** which may be either **live** or **dead**. States of all cells give a function $f : \mathbf{Z}^2 \rightarrow 2$ with

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \text{ is live;} \\ 0 & \text{if } (x, y) \text{ is dead.} \end{cases}$$

Figures such as



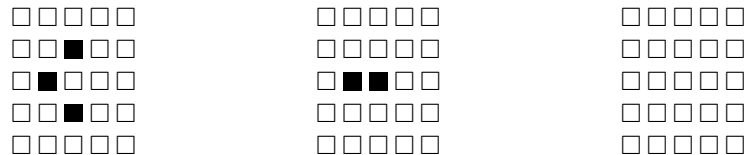
with \blacksquare live and \square dead, depict small collections of cells. Cell states evolve simultaneously and in discrete time intervals: if

$$\begin{array}{ccc} 5 & 6 & 7 \\ 3 & \circ & 4 \\ 0 & 1 & 2 \end{array}$$

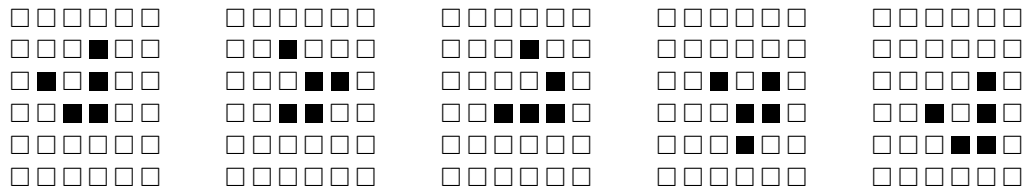
denotes a cell and its eight neighbors and L is the number of live adjacent cells then

$$\text{the subsequent state of cell } \circ = \begin{cases} \text{the current state of } \circ & \text{when } L = 2; \\ 1 & \text{when } L = 3; \\ 0 & \text{otherwise.} \end{cases}$$

The figures



illustrate evolution of a few cells, for example. A **glider** is a useful structure which evolves through four shapes while moving down and to the right:



[†] See *Autonomic Computing: I.B.M.'s Perspective on the State of Information Technology* at <http://www.ibm.com> and "Using Humans as a Computer Model" by Steve Lohr in the 15 October issue of *The New York Times*.

Conway's game is a functor $L : (\mathbf{N}, 0, +) \rightarrow \text{Set}$ with $L(*)$ the set of functions $[\mathbf{Z}^2, 2]$. A formula for the up-date rule $L(1) : [\mathbf{Z}^2, 2] \rightarrow [\mathbf{Z}^2, 2]$ appears in [3.27].

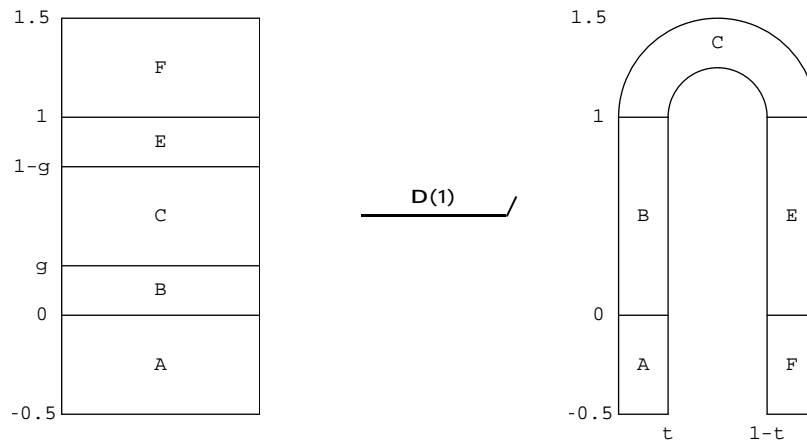
1.556. Motivated by problems in ordinary differential equations, Stephen Smale showed in 1963 that most diffeomorphisms of a manifold having dimension greater than one possess infinitely many periodic points and a richly-structured invariant set [Sm], [Wi]. Here is a family of examples which, though not diffeomorphisms, enjoy properties Smale described. For each pair (t, g) of parameters with $0 < t < \frac{1}{2}$ and $0 < g < \frac{1}{2}$ define a functor D from the monoid $(\mathbf{N}, 0, +)$ to Top, the category of topological spaces [III.2], by

$$D(*) = (x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq 1, -\frac{1}{2} \leq y \leq \frac{3}{2}$$

equipped with the subspace topology;

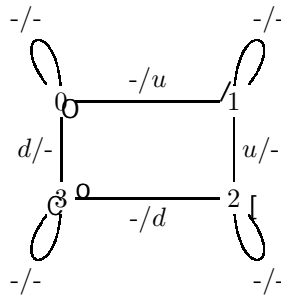
$$D(1)(x, y) = \begin{cases} (tx, y) & \text{if } y < 0; \\ tx, \frac{1}{g}y & \text{if } 0 \leq y \leq g; \\ \frac{1}{2} - \frac{1}{2} - tx \cos \frac{(y-g)}{1-2g}, 1 + \frac{1}{2} - tx \sin \frac{(y-g)}{1-2g} & \text{if } g < y < 1-g; \\ 1-tx, \frac{1}{g}(1-y) & \text{if } 1-g \leq y \leq 1; \\ (1-tx, 1-y) & \text{if } y > 1. \end{cases}$$

The figure below illustrates the action of $D(1)$.

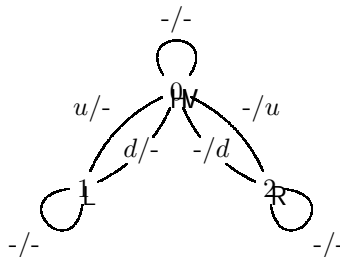


1.557. “Dining philosophers” refers to a simple, distributed, resource-allocation problem. A group of n individuals is seated for a meal at a round table. One chopstick is placed between each pair of diners who must eat according to the following protocol: first pick up a chopstick with the right hand; next with the left; after eating, put down the right chopstick first; finally the left. At each stage a diner may rest or wait while others are active. Two or more diners may act simultaneously. The following graph illustrates the states and

actions of a diner.



A diner holding no chopsticks is in state 0. To eat, one must be in state 2. States and transitions of a chopstick are shown below.



These two graphs are from [KSW1] where they are 1-cells in the bicategory $\text{Span}(\text{Graph})$. In [KSW1], coordination between a diner and an adjacent chopstick is modeled by composition of 1-cells and the entire system is a large, labeled graph. No nontrivial transitions are allowed if each philosopher has a chopstick in his or her right hand. The challenge to the group is avoiding deadlock.

A system of two diners and two chopsticks has eight states.



The protocol is the relation between states tabulated below. The state at the top of a column may transition to the state labeling a row iff the row-column location is filled with the symbol \circ .

	$\bullet = \bullet$	$\bullet - \bar{\bullet}$	$\bullet \bar{\bullet}$	$\bullet - \underline{\bullet}$	$\underline{\bullet} - \bullet$	$\bar{\bullet} \bullet$	$\bar{\bullet} - \bullet$	$\underline{\bullet} \bar{\bullet}$
$\bullet = \bullet$	\circ			\circ			\circ	
$\bullet - \bar{\bullet}$	\circ	\circ						
$\bullet \bar{\bullet}$		\circ	\circ					
$\bullet - \underline{\bullet}$			\circ	\circ				
$\underline{\bullet} - \bullet$	\circ				\circ			
$\bar{\bullet} \bullet$					\circ	\circ		
$\bar{\bullet} - \bullet$						\circ	\circ	
$\underline{\bullet} \bar{\bullet}$	\circ	\circ			\circ			\circ

The dining philosophers problem is a functor $A : (\mathbf{N}, 0, +) \rightarrow \text{Rel}$, with Rel the category composed of binary relations between sets [II.2]. $A(*)$ is the set of admissible states while $A(1) : A(*) \rightarrow A(*)$ is the relation

defined by the protocol:

$$(x, x^0) \in A(1) \text{ iff the system may transition from } x \text{ to } x^0.$$

1.558. Consider the following situation[†]. Three numbered marbles are placed in two boxes, A and B . Each second a random number between 1 and 3 is selected and the corresponding ball is moved from its current box to the other box. States of the system make up the set

$$S = \{AAA, AAB, ABA, BAA, ABB, BAB, BBA, BBB\}.$$

ABB is the state with marble 1 in box A and the other two in box B . If at one moment the system has state AAA , it can not have state BAB a second later since only one marble will have been transferred. Assuming no bias in marble-number selection, the probability is $1/3$ that the system will transition, for example, from AAB to BAB . Probabilities of all transitions are tabulated below.

	AAA	AAB	ABA	BAA	ABB	BAB	BBA	BBB
AAA :	0	$1/3$	$1/3$	$1/3$	0	0	0	0
AAB :	$1/3$	0	0	0	$1/3$	$1/3$	0	0
ABA :	$1/3$	0	0	0	$1/3$	0	$1/3$	0
BAA :	$1/3$	0	0	0	0	$1/3$	$1/3$	0
ABB :	0	$1/3$	$1/3$	0	0	0	0	$1/3$
BAB :	0	$1/3$	0	$1/3$	0	0	0	$1/3$
BBA :	0	0	$1/3$	$1/3$	0	0	0	$1/3$
BBB :	0	0	0	0	$1/3$	$1/3$	$1/3$	0

Each entry gives the probability of a transition from the state labeling its column to the state labeling its row. A single transition is a model of the partially ordered set $\bullet \text{---} \swarrow \bullet$ in the category Mes [1.89], [IV.3], [G]. Each node is interpreted as the measurable space $\mathcal{S} = (S, \mathcal{P}(S))$, with $\mathcal{P}(S)$ the power set (collection of all subsets) of S . The arrow is a morphism in Mes from \mathcal{S} to itself and is represented by the table.

1.559. A category \mathcal{J} is **small** if its class of morphisms is a set. For a category \mathcal{C} and a small category \mathcal{J} , $\mathcal{C}^{\mathcal{J}}$ is the category having functors $\mathcal{J} \rightarrow \mathcal{C}$ as objects and natural transformations as morphisms. $\mathcal{C}^{\mathcal{M}}$ with \mathcal{M} a monoid and $\mathcal{C}^{\mathcal{P}}$ with \mathcal{P} a poset are examples and inspire the slogan:

Categories of dynamic systems occur as functor categories.

Domain (syntax) and codomain (semantics) specify the two parts of the classification [1.55].

[†] Karl Peterson [P] introduced this situation, and attributed it to Ehrenfests, to illustrate the Poincare recurrence theorem.

1.6. Structures and representations

Constructions in category theory are specified using the available ingredients: morphisms and composition. Because of this, theorems keeping careful accounts of morphisms are significant [1.623]. This section gives the flavor of categorical definitions. See Appendix I and [BW2], [B], [Ma2], [FS].

1.61. A set 1 with one member is characterized up to bijection by the fact that for any set[†] X there is exactly one function $X \rightarrow 1$. Generalizing, an object 1 of a category \mathcal{C} is a **terminator** [I.42] if for any \mathcal{C} -object X there is a unique morphism $X \rightarrow 1$. A category $\text{Set}^{\mathcal{M}}$ of monoid actions, for example, has

$$\bullet \text{d}$$

as terminator.

In the category of sets, functions $1 \rightarrow X$ name members of X . For a category \mathcal{C} with terminator 1 , morphisms $1 \rightarrow X$ are **points** of X . In $\text{Set}^{\mathcal{M}}$, these name fixed-points.

1.611. A function

$$X \xrightarrow{f} Y$$

is injective iff $x = x^0$ whenever $f(x) = f(x^0)$. That is, f is injective iff

$$1 \begin{array}{c} \text{x} \\ \text{+} \\ \text{x}^0 \end{array} \xrightarrow{f} Y \quad \text{implies} \quad 1 \begin{array}{c} \text{x} \\ \text{+} \\ \text{x}^0 \end{array} \xrightarrow{f} Y.$$

This is equivalent to

$$X^0 \begin{array}{c} \text{x} \\ \text{+} \\ \text{x}^0 \end{array} \xrightarrow{f} Y \quad \text{implies} \quad X^0 \begin{array}{c} \text{x} \\ \text{+} \\ \text{x}^0 \end{array} \xrightarrow{f} Y$$

with X^0 any set, not just a singleton. A morphism $f : X \rightarrow Y$ in a category \mathcal{C} is **monic** [I.14] when

$$X^0 \begin{array}{c} \text{x} \\ \text{+} \\ \text{x}^0 \end{array} \xrightarrow{f} Y \quad \text{implies} \quad X^0 \begin{array}{c} \text{x} \\ \text{+} \\ \text{x}^0 \end{array} \xrightarrow{f} Y.$$

In categories of topological and measurable spaces, for example, monics are continuous, injective functions and measurable, injective functions respectively [III.2], [III.4]. In $\text{Set}^{\mathcal{M}}$ they are injective, equivariant maps.

1.612. For sets $X^0 \subset X$, the inclusion $X^0 \rightarrow X$ is injective. A categorical version can not rely on membership. A **subobject** [I.143] of an object X in a category \mathcal{C} is an equivalence class of monics with codomain X ; two monics being equivalent when there is an isomorphism φ for which

$$\begin{array}{ccc} X^0 & \xrightarrow{\quad} & X \\ \downarrow \varphi & \text{~~~~~} & \downarrow \text{id} \\ X^{00} & & X \end{array}$$

[†] Note the scope of this quantifier.

In the category of topological spaces, subobjects are (isomorphic to) subsets equipped with subspace topologies. In $\text{Set}^{\mathcal{M}}$ they are invariant subsets and have induced actions.

1.613. The cartesian product of sets X and Y comes equipped with projection maps

$$X \xrightarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y.$$

Moreover, functions $Z \rightarrow X \times Y$ are uniquely determined from pairs of functions $Z \rightarrow X$ and $Z \rightarrow Y$. Characterization of \wedge in predicate calculus is similar: proofs of $P \wedge Q$ give proofs of P and of Q and conversely, proofs of both give a proof of the conjunction. A **product** [I.43] of objects X and Y in a category \mathcal{C} is an object $X \times Y$ and a pair of morphisms

$$X \xrightarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

for which

$$\begin{array}{ccc} & Z & \\ \swarrow \text{F} & & \searrow \text{F} \\ X & \xrightarrow{\pi_1} X \times Y \xrightarrow{\pi_2} & Y \end{array} \quad \text{implies} \quad \begin{array}{ccc} & Z & \\ \swarrow \text{F} & \cdots & \searrow \text{F} \\ X & \xrightarrow{\pi_1} X \times Y \xrightarrow{\pi_2} & Y \end{array}$$

where the dotted arrow indicates existence of a unique morphism satisfying the implied commutativity conditions. The state-space of a product of two $\text{Set}^{\mathcal{M}}$ -objects X and Y is the cartesian product of $X(*)$ and $Y(*)$, for example.

1.614. Solving $x^2 - 1 = 0$ over the reals involves finding the largest $X^0 \subset \mathbb{R}$ for which

$$X^0 \xrightarrow{\pi_1} \mathbb{R} \xrightarrow{\pi_2} \mathbb{R} \quad \begin{array}{c} x^2 - 1 \\ + \\ 0 \end{array}$$

Any smaller set of solutions X^{00} (say, $X^{00} = \{1\}$) is included in $X^0 = \{-1, 1\}$ as asserted by:

$$\begin{array}{ccc} X^{00} & \cdots & \\ \vdots & \searrow & \\ X^0 & \xrightarrow{\pi_1} \mathbb{R} \xrightarrow{\pi_2} & \mathbb{R} \end{array} \quad \begin{array}{c} x^2 - 1 \\ + \\ 0 \end{array}$$

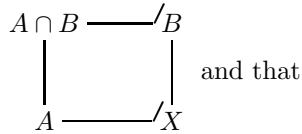
where the dotted arrow implies existence of a unique map satisfying the implied commutativity condition.

An **equalizer** [I.44] of morphisms $X \xrightarrow{f} Y$ in a category \mathcal{C} is a pair (X^0, e) for which

$$X^0 \xrightarrow{e} X \xrightarrow{\pi_1} X \xrightarrow{\pi_2} Y \quad \begin{array}{c} f \\ + \\ g \end{array}$$

and a universal property (analogous to that for solving equations) holds.

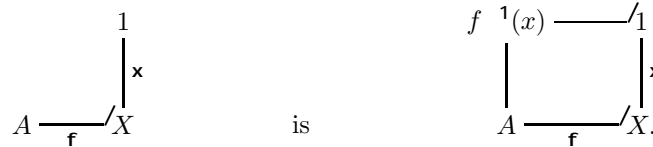
1.615. With sets $A \subset X$ and $B \subset X$ are associated inclusions $\begin{array}{c} B \\ | \\ A \xrightarrow{\quad} X \end{array}$. Intersection is characterized by



where the dotted arrow indicates existence of a unique map satisfying the implied commutativity conditions.

A **pullback** [I.45] of morphisms $\begin{array}{c} B \\ | \\ A \xrightarrow{f} X \end{array}$ in a category \mathcal{C} is a triple (P, f^0, g^0) for which $\begin{array}{c} P \xrightarrow{f^0} B \\ | \qquad \qquad | \\ A \xrightarrow{f} X \end{array}$ g^0

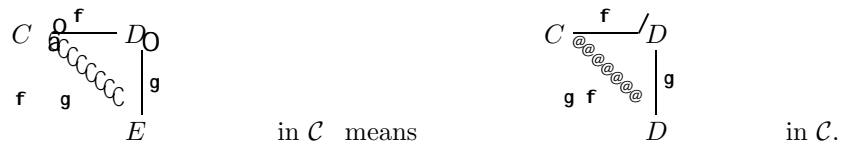
and a universal property (analogous to the one for above for intersection) holds. In the category of sets, for example, a pullback of



1.616. The **dual** [I.12], [I.22], [I.32] of a construction in the language of category theory is obtained by reversing arrows. The dual category \mathcal{C}^* of a category \mathcal{C} , for example, has an object C^* for each $C \in |\mathcal{C}|$, a morphism

$$C^* \xrightarrow{f^*} D^* \quad \text{for each} \quad C \xrightarrow{f} D \quad \text{in } \mathcal{C},$$

and composition \circ for which



$X \xrightarrow{f} Y$ is **epic**, the dual of **monic**, if

$$X \xrightarrow{f} Y \xrightarrow{+} Z \quad \text{implies} \quad X \xrightarrow{f} Y \xrightarrow{*} Z^*$$

Coproducts [I.53] of objects are dual to products and generalize disjoint unions of sets.

A **coequalizer** of morphisms $X \begin{matrix} \xrightarrow{f} \\ \oplus \\ \xrightarrow{g} \end{matrix} Y$ is a pair (Y^0, k) for which

$$X \begin{matrix} \xrightarrow{f} \\ \oplus \\ \xrightarrow{g} \end{matrix} Y \xrightarrow{k} Y^0$$

and a universal property holds. These generalize quotient operations [I.54].

1.62. A functor $U : \mathbf{Vect} \rightarrow \mathbf{Set}$ arises from forgetting the extra structure in the domain category. Let \mathcal{V} and \mathcal{W} be vector spaces. A basis X of \mathcal{V} is a useful gadget because of the bijection between **functions** $X \rightarrow U(\mathcal{W})$ and **linear transformations** $\mathcal{V} \rightarrow \mathcal{W}$: for a function $f : X \rightarrow U(\mathcal{W})$ there is a unique linear transformation $\mathbf{j} : \mathcal{V} \rightarrow \mathcal{W}$ for which $U(\mathbf{j}) \circ \eta = f$ where $\eta : X \rightarrow U(\mathcal{V})$ is inclusion. In diagrams:

$$\begin{array}{ccc} & & X \begin{matrix} \xrightarrow{\text{SSSSSSSSSS}} \\ \text{f} \\ \xrightarrow{\text{SSSSSSSSSS}} \end{matrix} U(\mathcal{W}). \\ & & \uparrow x \\ \mathcal{W} \xrightarrow{\Omega} \mathcal{V} & & U(\mathcal{V}) \xrightarrow{\text{u}(\mathbf{j})} U(\mathcal{W}). \\ & & \downarrow \mathbf{j} \\ & & \mathbf{Vect} & & \mathbf{Set} \end{array}$$

1.621. This is a special case of a general construction: a **reflection** [I.6] of an object B along a functor $G : \mathcal{A} \rightarrow \mathcal{B}$ is a pair $(F(B), \eta_B)$ with $F(B) \in |\mathcal{A}|$ and $B \xrightarrow{B} G(F(B))$ such that for any pair (A, b) with $A \in |\mathcal{A}|$ and $B \xrightarrow{b} G(A)$, there is a unique $F(B) \xrightarrow{a} A$ for which $G(a) \circ \eta_B = b$:

$$\begin{array}{ccc} & & B \begin{matrix} \xrightarrow{\text{SSSSSSSSSS}} \\ \text{B} \\ \xrightarrow{\text{SSSSSSSSSS}} \end{matrix} G(A). \\ & & \uparrow B \\ A \xrightarrow{\Omega} F(B) & & G(F(B)) \xrightarrow{G(a)} G(A). \\ & & \downarrow \eta_B \\ & & \mathcal{A} & & \mathcal{B} \end{array}$$

1.622. Dually, a **coreflection** of an object A along $F : \mathcal{B} \rightarrow \mathcal{A}$ is a pair $(G(A), \varepsilon_A)$ with $G(A) \in |\mathcal{B}|$ and $F(G(A)) \xrightarrow{\varepsilon_A} A$ such that for any pair (B, a) with $B \in |\mathcal{B}|$ and $F(B) \xrightarrow{a} A$, there is a unique $G(A) \xrightarrow{b} B$ such that $\varepsilon_A \circ F(b) = a$:

$$\begin{array}{ccc} & & A \begin{matrix} \xrightarrow{\text{SSSSSSSSSS}} \\ \varepsilon_A \\ \xrightarrow{\text{SSSSSSSSSS}} \end{matrix} F(B). \\ & & \uparrow \varepsilon_A \\ B \xrightarrow{\Omega} G(A) & & F(G(A)) \xrightarrow{F(b)} F(B). \\ & & \downarrow F \\ & & \mathcal{B} & & \mathcal{A} \end{array}$$

Exponentials (function spaces) are important examples. See 3.1.6 in Volume I of [B].

1.623. If a reflection along $G : \mathcal{A} \rightarrow \mathcal{B}$ exists for each object of \mathcal{B} , the objects $F(B)$ are part of a functor $\mathcal{B} \rightarrow \mathcal{A}$. Functors related in this way are **adjoints** [I.66] with F **left adjoint** to G and G **right adjoint** to F , denoted $F \dashv G$. On page 107 of [Ma2], Saunders Mac Lane explained that:

“The multiple of examples, here and elsewhere, of adjoint functors tend to show that adjoints occur almost everywhere in many branches of Mathematics. It is the thesis of this book that a systematic use of all these adjunctions illuminates and clarifies these subjects.”

Jim Lambek and P. J. Scott asserted that[†]

“Perhaps the most important concept which category theory has helped to formulate is that of adjoint functors.”

Terminators, equalizers, products, and pullbacks are instances of **limits** [I.411] in a category. The dual concepts are examples of **colimits**. Reasons behind the observations cited above are:

- Adjunctions provide bijections between sets of morphisms in **different** categories;
- A functor which has a left adjoint preserves all (small) limits[‡];
- Right adjoints preserve (small) colimits.

A representation $\mathcal{C}^{\mathcal{J}} \rightarrow \mathcal{C}^{\mathcal{J}^0}$ between categories of dynamic systems which has a left adjoint, for example, preserves invariant subobjects.

1.63. In [L4] and [L6], Lawvere interpreted the significance of certain adjoints occurring in the context of dynamic systems. These are cultivated further in [1.8] and Chapter 3 of this work. The following general constructions are used.

1.631. $F : \mathcal{A} \rightarrow \mathcal{B}$ with \mathcal{A} and \mathcal{B} small is a **small** functor. For a category \mathcal{C} , a small functor $F : \mathcal{A} \rightarrow \mathcal{B}$ induces $\mathcal{C}^{\mathbf{F}} : \mathcal{C}^{\mathcal{B}} \rightarrow \mathcal{C}^{\mathcal{A}}$ via $H \mapsto H \circ F$ on objects. For a morphism $\tau : H \Rightarrow H^0 : \mathcal{B} \rightarrow \mathcal{C}$, $\mathcal{C}^{\mathbf{F}}(\tau)_{\mathbf{A}} = \tau_{\mathbf{F}(\mathbf{A})}$ [1.8.1]–[1.8.8]. A reflection of $X : \mathcal{B} \rightarrow \mathcal{C}$ along $\mathcal{C}^{\mathbf{F}}$ is a **left Kan extension**. A coreflection is a **right Kan extension**.

1.632. Let \mathcal{J} be a small category. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ induces $F^{\mathcal{J}} : \mathcal{A}^{\mathcal{J}} \rightarrow \mathcal{B}^{\mathcal{J}}$ via $G \mapsto F \circ G$ on objects. For a morphisms $\tau : H \Rightarrow H^0 : \mathcal{J} \rightarrow \mathcal{A}$, $F^{\mathcal{J}}(\tau)_{\mathbf{A}} = F(\tau_{\mathbf{A}})$ [1.8(10)].

1.7. Program

In 1900 David Hilbert posed, with optimism for the future of his science, twenty-three problems which still challenge mathematicians [H]. The sixth involves a mathematical treatment of the axioms of physics and was influenced by his success in geometry. Contributing to the solution of this problem, Walter Noll laid down axioms in 1958 for a general theory describing materials which make up natural bodies [N1], [Tr1] and had revised this work by 1972 [N2]. Lawvere’s 1967 lectures (published in 1979 [L3]) outlined his “program to

[†] See page 12 of [LS].

[‡] See 1.83 in [FS].

(3) axiomatize the foundations of continuum mechanics in the spirit of Walter Noll on the basis of (2) a direct axiomatization of the essence of differential topology using results and methods of the French work in algebraic geometry . . . this requires (1) axiomatic study of categories of smooth sets, similar to the toposes of Grothendieck, since the most natural form of (2) is incompatible with usual set theory.”

1.711. In rational mechanics [Tr1], [Tr3], [TM], a **body** is to be distinguished from its shapes or placements in space. A **constitutive relation** is a function which determines the stresses **in** the body from the motion **of** the body. Noll’s theory includes restrictions on the form of these functions [Chapter IV of Tr1]:

“**Principle of Determinism:** The stress at a place occupied by the body-point X at time t is determined by the history of the motion of the body up to the time t .

“**Principle of Local Action:** The motion of body-points at a finite distance from X in some shape of the body may be disregarded in calculating the stress at X .

“**Principle of Material Frame-Indifference:**” expresses a kind of invariance under changes of space and time coordinates.

1.711. Expressing these in categorical language requires two more constructions. For a topological space \mathcal{X} , the poset of open sets ordered by inclusion is a small category, $\Omega(\mathcal{X})$. This gives the object part of a functor $\text{Top} \rightarrow \text{PoSet}$ [page 39 of J].

Let (P, \leq) be a poset. $P^0 \subset P$ is a **downdeal** if $x \in P^0$ whenever there is a $y \in P^0$ for which $x \leq y$. The set of downdeals is a topology on P This gives the object part of a functor $\text{PoSet} \rightarrow \text{Top}$ [page 22 of FS].

1.712. George H. Handelman contributed to [LSH] a chapter on one-dimensional, longitudinal motion of a bar. His tool for clarifying general concepts of continuum mechanics introduced me to the subject. $x \in \mathbf{R}$ is a position along the bar’s axis. The motion is to be described at each time $t \in \mathbf{R}^+$. $\sigma(x, t)$ and $\rho(x, t)$ respectively denote cross-sectional area and mass density. $u(x, t)$ is the displacement of the material at x from its initial position. Based on a few natural assumptions,

$$(\rho \sigma)_{\mathbf{t}} + (\rho \sigma u_{\mathbf{t}})_{\mathbf{x}} = 0$$

expresses conservation of mass (with $g_{\mathbf{x}}$ denoting $\frac{\partial g}{\partial \mathbf{x}}$ and $g_{\mathbf{t}} = \frac{\partial g}{\partial \mathbf{t}}$ for a differentiable $g(x, t)$). Two types of forces act on the bar: $T(x, t)$ is the stress (force per unit area) exerted **by** material left of x **on** material right of x ; $f(x, t)$ is body force per unit mass. Momentum balance is

$$\rho \sigma (u_{\mathbf{t}\mathbf{t}} + u_{\mathbf{t}} u_{\mathbf{t}\mathbf{x}}) = \rho \sigma f + (T \sigma)_{\mathbf{x}} .$$

The stress resulting from a given deformation depends on the material composing the bar. A linearly-elastic solid has

$$T(x, t) = E(x, t) u_{\mathbf{x}}(x, t),$$

with $E(x, t)$ Young's modulus, in which case momentum balance becomes:

$$\rho \sigma (u_{tt} + u_t u_{tx}) = \rho \sigma f + (E u_x \sigma)_x .$$

Modeling the bar involves making assumptions about smoothness of material properties variations. Given a time t , let $t_{\#}$ be the set of instants prior to t . Let $U \subset \mathbf{R}$ be open (in the usual topology). Giving the set of admissible deformations in U prior to t as

$$M(t_{\#})(U) = (\rho, \sigma, u) : U \times t_{\#} \rightarrow \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R} \mid \rho, \sigma, u \in C^1$$

asserts that density, area, and displacement vary smoothly. If the admissible stresses and body forces are

$$T(t_{\#})(U) = \{ \tau : U \times t_{\#} \rightarrow \mathbf{R} \mid \tau \in C^1 \}$$

and

$$F(t_{\#})(U) = \{ \varphi : U \times t_{\#} \rightarrow \mathbf{R} \mid \varphi \in C^1 \}$$

then these quantities do as well. The notation indicates that M , T and F are functors

$$\Omega(D(\mathbf{R}^+, \leq)) \longrightarrow \text{Set}^{(\mathbf{R})} ,$$

that is, objects of a particular functor category. Mass balance extracts a subobject

$$M^0(t_{\#})(U) = \{ (\rho, \sigma, u) \in M(t_{\#})(U) \mid (\rho \sigma)_t + (\rho \sigma u_t)_x = 0 \}$$

of M and gives an inclusion

$$M^0 \xrightarrow{\text{mass balance}} M .$$

That body force may be computed from knowledge of the motion and stress by

$$f = u_{tt} + u_t u_{tx} - \frac{(T \sigma)_x}{\rho \sigma}$$

gives a natural transformation[†]

$$M \times T \xrightarrow{\text{momentum balance}} F .$$

The constitutive relation gives $\kappa : M \longrightarrow T$ via

$$\kappa_{t_{\#} \mathbf{u}} (\rho, \sigma, u) = E u_x$$

assuming E (part of the definition of κ) is C^1 . These ingredients give

$$M^0 \xrightarrow{\text{mass balance}} M \xrightarrow{\text{id; } (\text{id}; \cdot)} M \times T \xrightarrow{\text{momentum balance}} F$$

[†] Products in such functor categories are computed pointwise [page 22 of MM].

Composition indicates that body force can be computed from the motion if the constitutive relation is known. Specifying a particular body force gives

$$\begin{array}{c}
 1 \\
 | \\
 M^0 \text{ --- } / F \text{ --- } | \mathbf{f}
 \end{array}$$

and pullback selects the motions compatible with f and κ :

$$\begin{array}{ccc}
 M^{00} & \text{---} & / 1 \\
 | & & | \mathbf{f} \\
 M^0 & \text{---} & / F.
 \end{array}$$

1.713. A continuous function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a local homeomorphism if each $x \in \mathcal{X}$ has an open neighborhood U for which the restriction of f to U is a homeomorphism onto an open subset of \mathcal{Y} .

A sheaf on a topological space \mathcal{Y} is a pair (\mathcal{X}, f) with \mathcal{X} a topological space and $f : \mathcal{X} \rightarrow \mathcal{Y}$ a local homeomorphism [page 12 of FS]. For fixed \mathcal{Y} , sheaves are objects of a category $\text{Sh}(\mathcal{Y})$: a morphism $(\mathcal{X}, f) \rightarrow (\mathcal{X}^0, f^0)$ is a continuous $h : \mathcal{X} \rightarrow \mathcal{X}^0$ for which

$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{h} & \mathcal{X}^0 \\
 @. @VV f V @VV f^0 V \\
 & & \mathcal{Y}.
 \end{array}$$

1.714. Let \mathcal{B} be a topological space, imagined as a physical body, let \mathcal{T} be a poset of times, and let \mathcal{C} be a category. A static constitutive specification of \mathcal{B} in \mathcal{C} is a subcategory \mathcal{P} of $\mathcal{C}^{(\mathcal{B})}$. A \mathcal{P} -object is a description of, for example, admissible placements or stresses. A subcategory \mathcal{A} of $\mathcal{P}^{(\mathcal{D}(\mathcal{T}))}$ is a dynamic constitutive specification. An \mathcal{A} -object is an activity and an \mathcal{A} -morphism is a constitutive relation. By definition, constitutive relations satisfy Determinism. A constitutive relation $\kappa : M \rightarrow T$ between activities also satisfies Local Action: for a downdeal $t_\#$, the component $\kappa_{t_\#} : M(t_\#) \rightarrow T(t_\#)$ is a natural transformation between functors $\Omega(\mathcal{B}) \rightarrow \mathcal{C}$. $U^0 \subset U$ in \mathcal{B} implies:

$$\begin{array}{ccc}
 M(t_\#)(U) & \xrightarrow{(\kappa_{t_\#})_U} & T(t_\#)(U) \\
 \mathbf{M}(t_\#)(\mathbf{U}^0 \ \mathbf{U}) \Big| & & \Big| \mathbf{T}(t_\#)(\mathbf{U}^0 \ \mathbf{U}) \\
 M(t_\#)(U^0) & \xrightarrow{(\kappa_{t_\#})_{U^0}} & T(t_\#)(U^0).
 \end{array}$$

Material Frame-Indifference might be formulated in terms of a specified class of endo-functors on $\mathcal{T} \times \mathcal{H}$. Calculations described on pages 165–176 of [L6] may have been done with Frame-Indifference in mind.

1.715. Using the language of category theory to investigate relationships between continuum and kinetic models of fluid flows requires categorical specifications of constitutive relations and of idioms occurring across

classifications of dynamic systems. The approach taken here is influenced by the “original feature” of [FS] and my ongoing studies of it[†].

For each category considered in this work I have pursued the following program. Use representations to:

1. characterize monics, epics, and isomorphisms;
2. determine which limits exist;
3. describe the structure of posets of subobjects;
4. determine if the category has images and factorization systems.

This program is incomplete in that I do not discuss 3. or 4. for any categories. In some cases, notably Mes and StM, 1. and 2. have gaps to be filled.

Let \mathcal{T} be a partially ordered set (of time instants). That

$$\text{Set}^{\mathbf{D}(\mathcal{T})} \cong \text{Sh}(D(\Omega(D(\mathcal{T}))))$$

[page 22 of FS] suggests the slogan:

Categories composed of constitutive relations may occur as categories of sheaves.

If \mathcal{T} is the ordered set of reals, positive reals, or an interval $[t_0, t_1]$, supremum gives a functor $s : \Omega(D(\mathcal{T})) \rightarrow \mathcal{T}$.

An activity which is a function of only time instants rather than time histories is one which factors through s :

$$\begin{array}{ccc} \Omega(D(\mathcal{T})) & \xrightarrow{\mathbf{A}} & \mathcal{H} \\ \downarrow s & \searrow \text{dotted} & \\ \mathcal{T} & & \end{array}$$

For such activities without memory, the isomorphism

$$\text{Set}^{\mathcal{T}} \cong \text{Sh}(D(\mathcal{T}))$$

reinforces the above slogan.

There is a functor $\delta^0 : (\mathbf{R}^+, \leq) \rightarrow (\mathbf{R}^+, 0+)$ via $\delta^0(t) = *$ and $\delta^0(t, t^0) = t^0 - t$. An activity is a **ow** if it factors through this functor as well:

$$\begin{array}{ccc} \Omega(D(\mathbf{R}^+, \leq)) & \xrightarrow{\mathbf{A}} & \mathcal{H} \\ \downarrow s & \searrow \text{dotted} & \downarrow \text{dotted} \\ (\mathbf{R}^+, \leq) & \xrightarrow{\delta^0} & (\mathbf{R}^+, 0, +) \end{array}$$

Representations among categories of monoid and poset actions are considered in this work.

[†] In his review of [FS], Walter Tholen wrote: “The pattern of their approach becomes clear right at the beginning of the book: derive representation theorems and thereby establish the connections to logic and geometry. [They] present category theory as a challenging mathematical subject, with (by comparison with other books on the subject) unprecedented rigour and brevity.”

1.72. Gaining understanding of the relationship between continuum and kinetic theories of gasses is a motivation throughout this work. It has been useful, however, to study how category theory is applied to other problems. Here are ideas for future projects.

1.721. Missing from this work are categorical formulations of stability [Wi] and asymptotics. These are required to describe one dynamic system as being a “relaxed” form of another [1.22], [1.882].

1.722. Can a useful theory of complexity of systems be developed using the structures of posets of subobjects? The dimension of a dynamic system, for example, could be defined as the length of the longest chain of subobjects.

1.723. Implementing algorithms for computing Kan extensions [CLW], [W] and calculating in the bicategory $\text{Span}(\text{Graph})$ [KSW1], [KSW2], [We] would provide me with deeper understanding of category theory and better programming skills. Solving real engineering problems using categorical formulations is likely to require such tools. These could also be useful teaching devices. I have funding for the summer of 2002 to direct an undergraduate research project. The goal will be to understand some simple, distributed resource allocation problems (dining philosophers, Peterson’s mutual exclusion algorithm) and develop Java applets illustrating what we learn.

1.724. Gary Sherman at Rose-Hulman has developed an active undergraduate research program during the past 20 years. He and his students study ‘CwatSets,’ structures arising in statistical inference and involving wreath products of groups. He hopes to find a categorical presentation of his work. Working on this issue would give me an opportunity to study fibrations and categorical wreath products [Chapter 12 of BW2].

1.725. A route to understanding logic and model theory is through the categorical approach [Cr], [FS], [Go], [Ja], [MP], [Mc], [T]. Can topos theory can be used to formulate, generalize, and implement machine learning algorithms [M]?

1.726. How can the category Mes [IV.3] be used to study particular examples of stochastic processes [Chapter 9 of F] and to formulate concepts in the theory of turbulent fluid flows [Bat1]?

1.727. “Tensors” occurring in continuum mechanics [Tr1] are sections of vector bundles [MM]. Can an understanding of the latter give useful information about constitutive relations or a formulation of Frame-Indifference?

1.728. The conjectures regarding weak products in Mes and functoriality of the Carathéodory outer measure construction should be resolved [1.892], [1.893].

1.729. Rediscovering basic properties of categories of manifolds could make useful undergraduate research projects. Over time this could develop into a program in which category theory is used to study current topics in physics [C], [FU].

1.72(10). What is categorical shape theory [CP] and how can it be applied to engineering problems?

1.8. Synopsis

Small functors [1.631] arising in the context of dynamic systems include:

$$\mathcal{M} \xrightarrow{!} \mathbb{1} \xrightarrow{\quad} \mathcal{M}$$

with \mathcal{M} a monoid and $\mathbb{1}$ a trivial monoid; an inclusion

$$P \xrightarrow{0} \mathcal{P}$$

with \mathcal{P} a poset and P its underlying set;

$$\begin{array}{ccccc} (\mathbb{N}, \leq) & \xrightarrow{\quad} & \mathbb{Z}(\mathbb{N}, 0, +) & \xrightarrow{\downarrow_n} & (\mathbb{N}_n, 0, +) \\ \downarrow^0 & & \downarrow & & \\ (\mathbb{R}^+, \leq) & \xrightarrow{0} & \mathbb{Z}(\mathbb{R}^+, 0, +) & \xrightarrow{\downarrow_x} & (\mathbb{R}_x^+, 0, +) \end{array}$$

with ι and ι^0 inclusions, δ and δ^0 differences: $(x, y) \mapsto y - x$; \downarrow_n projection onto the positive integers mod(n), and \downarrow_x also a projection;

$$(\mathbb{R}^+, 1, \times) \xrightarrow{(\)^x} (\mathbb{R}^+, 1, \times) \qquad (\mathbb{N}, 1, \times) \xrightarrow{(\)^n} (\mathbb{N}, 1, \times)$$

with $(\mathbb{R}^+, 1, \times)$ and $(\mathbb{N}, 1, \times)$ multiplicative monoids and $(\)^y$ exponentiation by a fixed power. These provide [1.632] representations among categories of dynamic systems:

$$\mathcal{C}^{\mathbb{M}} \xrightarrow{\mathbf{0} \ T} \mathcal{C} \xrightarrow{\mathbf{0} \ \text{ev}} \mathcal{C}^{\mathbb{M}} \qquad \mathcal{C}^{\mathbb{P}} \xrightarrow{\mathbf{0} \ C^{\eta^0}} \mathcal{C}^{\mathbb{P}}$$

with $T = \mathcal{C}^{\mathbb{1}} \circ \Delta$, $\text{ev} = \text{ev}_? \circ \mathcal{C}$, and $\Delta : \mathcal{C} \rightarrow \mathcal{C}^{\mathbb{1}}$ an isomorphism with inverse $\text{ev}_?$;

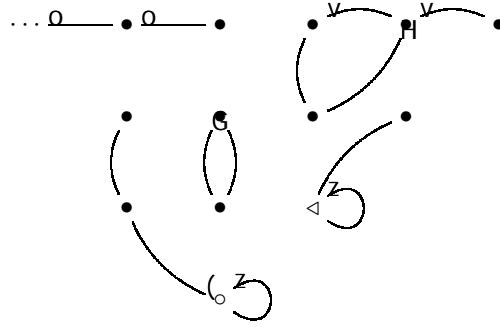
$$\begin{array}{ccccc} \mathcal{C}^{(\mathbb{N}; \)} & \xrightarrow{\mathbf{0} \ C^\delta} & \mathcal{C}^{(\mathbb{N}; \mathbb{0} \ +)} & \xrightarrow{\mathbf{0} \ C^{\natural_n}} & \mathcal{C}^{(\mathbb{N}_n; \mathbb{0}; \ +)} \\ \downarrow^{C^{\iota^0}} & & \downarrow^{C^\iota} & & \\ \mathcal{C}^{(\mathbb{R}^+; \)} & \xrightarrow{\mathbf{0} \ C^{\delta^0}} & \mathcal{C}^{(\mathbb{R}^+; \mathbb{0}; \ +)} & \xrightarrow{\mathbf{0} \ C^{\natural_x}} & \mathcal{C}^{(\mathbb{R}_x^+; \mathbb{0}; \ +)} \end{array}$$

$$\mathcal{C}^{(\mathbb{R}^+; 1; \)} \xrightarrow{\mathbf{0} \ C^{(\)^x}} \mathcal{C}^{(\mathbb{R}^+; 1; \)} \qquad \mathcal{C}^{(\mathbb{N}; 1; \)} \xrightarrow{\mathbf{0} \ C^{(\)^n}} \mathcal{C}^{(\mathbb{N}; 1; \)} .$$

Lawvere and others have given interpretations of these and their adjoints [La4], [La6], [CLW]. In his review of [L3], Gavin Wraith wrote: “Like most of [Lawvere’s] articles, it is very compactly written, a distillation and progress report on a large program.” For this reason it is difficult to deduce which of the various conditions on a category \mathcal{C} ensuring existence of adjoints[†] for the above representations are restatements of unpublished results.

[†] See [3.131], [3.141], [3.231], [3.241], [3.342], [3.351], [3.433], [3.443], [3.533], [3.543], [3.633], [3.643], [3.732], [3.742], [3.833], and [3.843].

1.81. $T : \mathcal{C} \rightarrow \mathcal{C}^M$ maps a \mathcal{C} -object C to an action with trivial dynamics [3.12]. A right adjoint [1.623] $\text{FP} : \text{Set}^M \rightarrow \text{Set}$ of T computes sets of fixed points [page 155 of L6]. The action $(\mathbf{N}, 0, +) \rightarrow \text{Set}$ shown below



for example, maps to the set $\{o, \triangleleft\}$, where different symbols indicate the image of ε [1.622]. A Smale horseshoe [1.556], viewed as a functor $(\mathbf{N}, 0, +) \rightarrow \text{Set}$, gives an isomorphic set: $(0, 0), \frac{1}{1+t}, \frac{1}{1+g}$. A periodic orbit

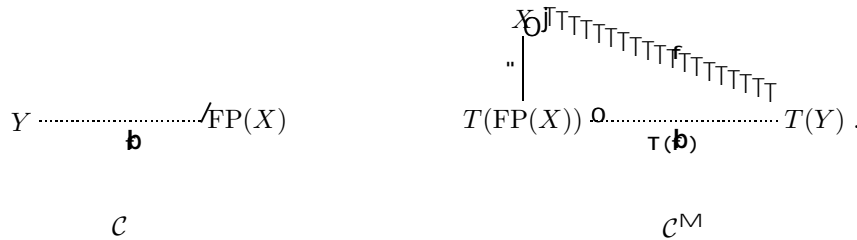


maps to the empty set while Conway's "Game of Life" [1.555] has a large set of time-invariant states. The flow $F : (\mathbf{R}^+, 0, +) \rightarrow \text{Set}$ having $F(*) = \mathbf{R}$ and induced by a logistic equation

$$y^0 = \alpha y \left(1 - \frac{y}{\beta}\right)$$

with $\alpha \neq 0$ becomes $\{0, \beta\}$.

1.811. The universal property relating T and FP is the following: for $X : \mathcal{M} \rightarrow \mathcal{C}$, there is an equivariant $\varepsilon : T(\text{FP}(X)) \Rightarrow X$ such that $Y \in |\mathcal{C}|$ and $f : T(Y) \Rightarrow X$ equivariant implies existence of a unique $\mathbf{p} \in \mathcal{C}(Y, \text{FP}(X))$ for which $\varepsilon \circ T(\mathbf{p}) = f$:

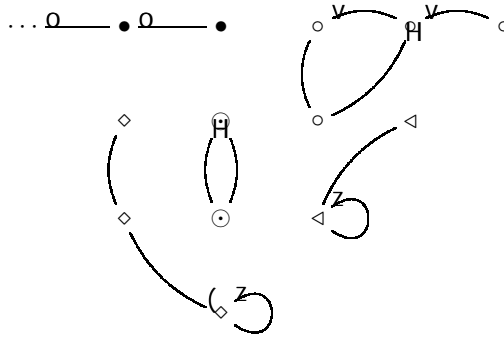


Theorem 3.131. If \mathcal{C} has equalizers and M -indexed products then $T : \mathcal{C} \rightarrow \mathcal{C}^M$ has a right adjoint.

A right adjoint may be computed using the equalizers and products described in the proof. Formulas with $\mathcal{C} = \text{Set}$ are listed in [3.132].

1.812. A left adjoint $\text{Orb} : \text{Set}^{\mathbf{M}} \rightarrow \text{Set}$ of T gives the “set of orbits” of an action [page 154 of L6], [CLW].

Orb maps the action $(\mathbf{N}, 0, +) \rightarrow \text{Set}$ depicted by

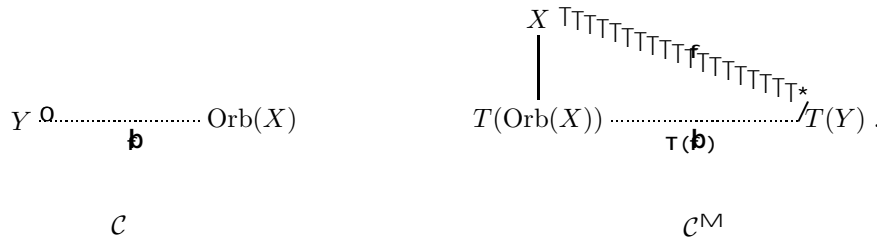


to the set $\{\diamond, \bullet, \odot, \triangleleft, \circ\}$ where different symbols indicate the image of η [1.621]. The flow $F : (\mathbf{R}^+, 0, +) \rightarrow \text{Set}$ obtained from

$$y^0 = \alpha y \quad 1 - \frac{y}{\beta}$$

maps to an isomorphic set.

1.813. The universal property relating T and Orb is the following: for $X : \mathcal{M} \rightarrow \mathcal{C}$ there is an equivariant $\eta : X \Rightarrow T(\text{Orb}(X))$ such that $Y \in |\mathcal{C}|$ and $f : X \Rightarrow T(Y)$ equivariant implies existence of a unique $\mathfrak{b} \in \mathcal{C}(\text{Orb}(X), Y)$ for which $T(\mathfrak{b}) \circ \eta = f$:



Theorem 3.141: If \mathcal{C} has coequalizers and M -indexed coproducts then T has a left adjoint.

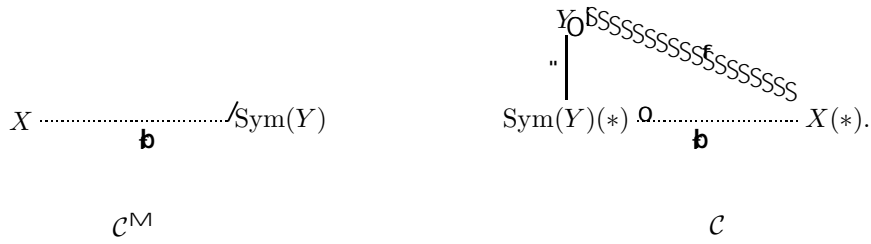
A left adjoint may be computed using the coequalizers and coproducts described in the proof. Formulas with $\mathcal{C} = \text{Set}$ are listed in [3.142].

1.82. Evaluation $\text{ev} : \mathcal{C}^{\mathbf{M}} \rightarrow \mathcal{C}$ produces the underlying object of a dynamic system [3.22]. Lawvere observed that a right adjoint $\text{Sym} : \mathcal{C} \rightarrow \mathcal{C}^{\mathbf{M}}$ of ev gives symbolic dynamics [L4]. The iterator obtained from $2 = \{0, 1\}$, for example, is the the set of sequences $\mathbf{N} \rightarrow 2$ equipped with left-shift dynamics:

$$011001 \dots \quad \mapsto \quad 11001 \dots$$

Right adjointness is expressed as follows: given an action $X : \mathcal{M} \rightarrow \mathcal{C}$ and a \mathcal{C} -object Y , for any \mathcal{C} -morphism

$f : X(*) \rightarrow Y$, there is a unique, equivariant $\mathfrak{p} : X \Rightarrow \text{Sym}(Y)$ for which $\varepsilon \circ \mathfrak{p} = f$:



Lawvere interpreted f as a **measurement** taken on the underlying space $X(*)$ of the action and described it as **chaotic** if Y is not terminal and the induced \mathfrak{p} is epic. He defined an action to be **chaotic** if it has an invariant subobject X^0 which may be equipped with a chaotic measurement [L4], [3.25].

Theorem 3.231: Let \mathcal{M} be a monoid. If \mathcal{C} has equalizers and M -indexed products then ev has a right adjoint. It need not be full.

A right adjoint may be computed using the equalizers and products described in the proof. Formulas with $\mathcal{C} = \text{Set}$ are listed in [3.233].

Theorem 3.26: Lawvere's de nition of chaotic dynamic system applies to the Smale horseshoe, viewed as a functor $(\mathbb{N}, 0, +) \rightarrow \text{Top}$.

Theorem 3.27: It also applies to Conway's \Game of Life."

The measurement used in the proof is defined inductively using the infinite grid on which the game is played.

1.821. Formulas for symbolic dynamics of non-deterministic systems $(\mathbb{N}, 0, +) \rightarrow \text{Rel}$ are listed in [3.235]. A system of two dining philosophers [1.557] may be equipped with the measurement $f : X \rightarrow \{D, N\}$ indicating the conditions of being in or out of deadlock. Symbolic dynamics gives the sequence of sets of measurements accessible from each initial state:

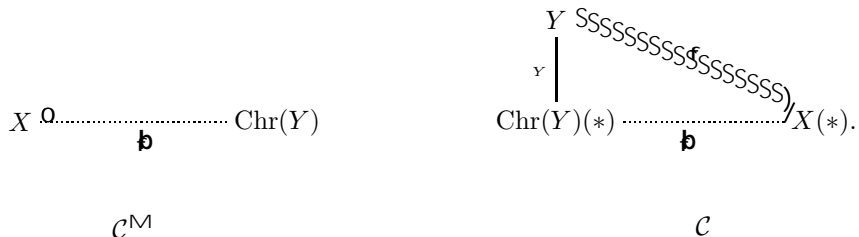
	0	1	2	3	
• = •	{N}	{D, N}	{D, N}	{D, N}	...
• — $\bar{\bullet}$	{N}	{D, N}	{D, N}	{D, N}	...
• — $\underline{\bullet}$	{N}	{N}	{N}	{D, N}	...
• — \bullet	{N}	{N}	{D, N}	{D, N}	...
$\underline{\bullet}$ — •	{N}	{D, N}	{D, N}	{D, N}	...
$\bar{\bullet}$ •	{N}	{N}	{N}	{D, N}	...
$\bar{\bullet}$ — •	{N}	{N}	{D, N}	{D, N}	...
$\underline{\bullet}$ $\bar{\bullet}$	{D}	{D}	{D}	{D}	...

Of particular interest are states, such as $\bullet = \bullet$, $\bullet - \bar{\bullet}$, and $\underline{\bullet} - \bullet$ in which deadlock is imminent.

1.822. A left adjoint $\text{Chr} : \text{Set} \rightarrow \text{Set}^{\mathcal{M}}$ computes an evolving chronicle of states, mapping a set Y to $\text{Chr}(Y) : \mathcal{M} \rightarrow \text{Set}$ with

$$\text{Chr}(Y)(*) = M \times Y \quad \text{and} \quad \text{Chr}(Y)(m) : M \times Y \rightarrow M \times Y \quad \text{via } (n, y) \mapsto (m * n, y).$$

The universal property relating Chr and ev is the following: for $Y \in |\mathcal{C}|$, there is a \mathcal{C} -morphism $\eta_Y : Y \rightarrow \text{Chr}(Y)(*)$ such that $X : \mathcal{M} \rightarrow \mathcal{C}$ and $f \in \mathcal{C}(Y, X(*))$ implies existence of a unique equivariant $\mathfrak{p} : \text{Chr}(Y) \Rightarrow X$ for which $\mathfrak{p} \circ \eta_Y = f$:



Theorem 3.241. If \mathcal{C} has M -indexed coproducts then $\text{ev} : \mathcal{C}^{\mathcal{M}} \rightarrow \mathcal{C}$ has a left adjoint $\text{Chr} : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{M}}$. It need not be full.

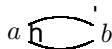
A left adjoint may be computed using the coequalizers and coproducts described in the proof. Formulas with $\mathcal{C} = \text{Set}$ are listed in [3.243].

1.83. By ignoring the dynamics, a poset action $\mathcal{P} \rightarrow \mathcal{C}$ may be viewed as merely a collection of state spaces $P \rightarrow \mathcal{C}$. This is the effect of $\mathcal{C}^0 : \mathcal{C}^{\mathcal{P}} \rightarrow \mathcal{C}^{\mathcal{P}}$ where $\eta^0 : P \rightarrow \mathcal{P}$ is inclusion.

Theorem 3.341. Let \mathcal{P} be a poset. If \mathcal{C} has P -indexed products then $\mathcal{C}^0 : \mathcal{C}^{\mathcal{P}} \rightarrow \mathcal{C}^{\mathcal{P}}$ has a right adjoint. Formulas with $\mathcal{C} = \text{Set}$ and $\mathcal{P} = (\mathbf{N}, \leq)$ are listed in [3.343]. The construction described in the proof of [3.334] motivates the calculations in [1.894].

Theorem 3.342. Let \mathcal{P} be a poset. If \mathcal{C} has P -indexed coproducts then $\mathcal{C}^0 : \mathcal{C}^{\mathcal{P}} \rightarrow \mathcal{C}^{\mathcal{P}}$ has a left adjoint. Universal properties characterizing right and left Kan extensions along $\eta^0 : P \rightarrow \mathcal{P}$ are listed in [3.34] and [3.35].

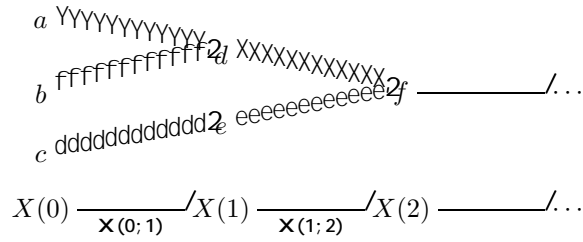
1.84. From an iterator $(\mathbf{N}, 0, +) \rightarrow \mathcal{C}$, the functor \mathcal{C} produces a transition $(\mathbf{N}, \leq) \rightarrow \mathcal{C}$ having time-invariant state space and transition rule [pages 158–162 of L6]. For example, $X : (\mathbf{N}, 0, +) \rightarrow \text{Set}$ depicted by



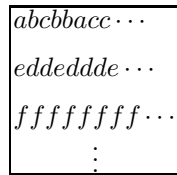
induces the functor $\text{Set}(X) : (\mathbf{N}, \leq) \rightarrow \text{Set}$ shown below

$$\{a, b\} \xrightarrow{\text{swap}} \{a, b\} \xrightarrow{\text{swap}} \dots$$

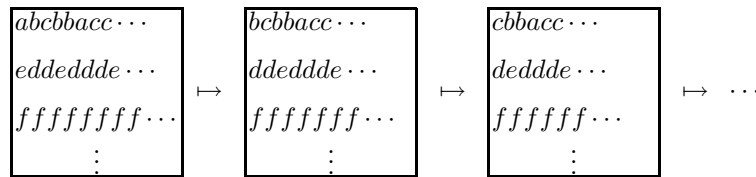
A right adjoint $\text{Ran} : \mathcal{C}^{\mathbb{N}; \cdot} \rightarrow \mathcal{C}^{\mathbb{N}; 0; +}$ produces iterators from transitions and may be viewed as a kind of symbolic dynamics. For the transition $X : (\mathbb{N}, \leq) \rightarrow \text{Set}$:



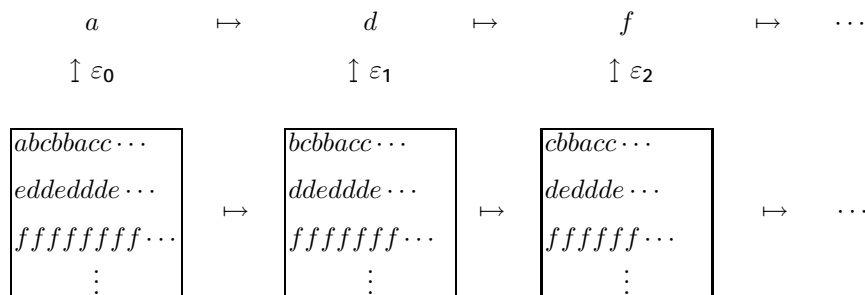
a typical member of $\text{Ran}(X)(*)$ is



in which the first row is a sequence $\mathbb{N} \rightarrow X(*)$, the second is its image $\mathbb{N} \xrightarrow{X(*)} X(*) \xrightarrow{x(0;1)} X(*)$, etc. Dynamics of $\text{Ran}(X)$ is left shift of each row:



Application of the equivariant $\varepsilon : \text{Ran}(X) \circ \delta \Rightarrow X$ is illustrated by



1.841. A right Kan extension of a transition $X : (\mathbb{N}, \leq) \rightarrow \mathcal{C}$ is an iterator $\text{Ran}(X) : (\mathbb{N}, 0, +) \rightarrow \mathcal{C}$ and equivariant $\varepsilon : \text{Ran}(X) \circ \delta \Rightarrow X$ with the following property: for each iterator $Y : (\mathbb{N}, 0, +) \rightarrow \mathcal{C}$ and

equivariant $f : Y \circ \delta \Rightarrow X$, there is a unique equivariant $\mathfrak{b} : Y \Rightarrow \text{Ran}(X)$ for which $\varepsilon \circ (\mathfrak{b} * id) = f$:

$$\begin{array}{ccc}
 Y & \xrightarrow{\quad \mathfrak{b} \quad} & \text{Ran}(X) \\
 \mathcal{C}^{(\mathbb{N}; 0; +)} & & \\
 & & \text{Ran}(X) \circ \delta \xrightarrow{\quad \mathfrak{b} id_\delta \quad} Y \circ \delta. \\
 & & \mathcal{C}^{(\mathbb{N}; \quad)}
 \end{array}$$

Theorem 3.433: If \mathcal{C} has equalizers and countable products then \mathcal{C} has a right adjoint.

A right adjoint may be computed using the equalizers and products described in the proof. Formulas with $\mathcal{C} = \text{Set}$ are listed in [3.434].

1.842. The universal property characterizing left Kan extensions along δ is listed in [3.44].

Theorem 3.443: If \mathcal{C} has coequalizers and countable coproducts then \mathcal{C} has a left adjoint.

1.85. From a flow $(\mathbb{R}^+, 0, +) \rightarrow \mathcal{C}$, the functor \mathcal{C}^0 produces a transition $(\mathbb{R}^+, \leq) \rightarrow \mathcal{C}$ having time-invariant state space and transition rule [pages 158–162 of L6], [3.5]. Universal properties characterizing left and right Kan extensions along δ^0 are listed in [3.53], [3.54]. Let \mathfrak{c} denote the cardinality of the set of real numbers.

Theorem 3.533: If \mathcal{C} has equalizers and \mathfrak{c} products then \mathcal{C}^0 has a right adjoint.

A right adjoint may be computed using the equalizers and products described in the proof. Formulas with $\mathcal{C} = \text{Set}$ are listed in [3.534].

Theorem 3.543: If \mathcal{C} has coequalizers and \mathfrak{c} coproducts then \mathcal{C}^0 has a left adjoint.

A left adjoint may be computed using the coequalizers and coproducts described in the proof. Formulas with $\mathcal{C} = \text{Set}$ are listed in [3.542].

1.86. From a flow $(\mathbb{R}^+, 0, +) \rightarrow \mathcal{C}$, the functor \mathcal{C} produces an iterator $(\mathbb{N}, 0, +) \rightarrow \mathcal{C}$ [page 149 of L6], [3.6]. This is the process of observing a continuous-time system only at discrete time intervals.

1.861. A right Kan extension along ι of an iterator $X : (\mathbb{N}, 0, +) \rightarrow \mathcal{C}$ is a flow $\text{Ran}(X) : (\mathbb{R}^+, 0, +) \rightarrow \mathcal{C}$ and equivariant $\varepsilon : \text{Ran}(X) \circ \iota \Rightarrow X$ with the following property: for each flow $Y : (\mathbb{R}^+, 0, +) \rightarrow \mathcal{C}$ and equivariant $f : Y \circ \iota \Rightarrow X$, there is a unique equivariant $\mathfrak{b} : Y \Rightarrow \text{Ran}(X)$ for which $\varepsilon \circ (\mathfrak{b} * id) = f$:

$$\begin{array}{ccc}
 Y & \xrightarrow{\quad \mathfrak{b} \quad} & \text{Ran}(X) \\
 \mathcal{C}^{(\mathbb{R}^+; 0; +)} & & \\
 & & \text{Ran}(X) \circ \iota \xrightarrow{\quad \mathfrak{b} id_\iota \quad} Y \circ \iota. \\
 & & \mathcal{C}^{(\mathbb{N}; 0; +)}
 \end{array}$$

Theorem 3.633: If \mathcal{C} has equalizers and c products then \mathcal{C} has a right adjoint.

A right adjoint may be computed using the equalizers and products described in the proof. Formulas for $\mathcal{C} = \text{Set}$ are listed in [3.634].

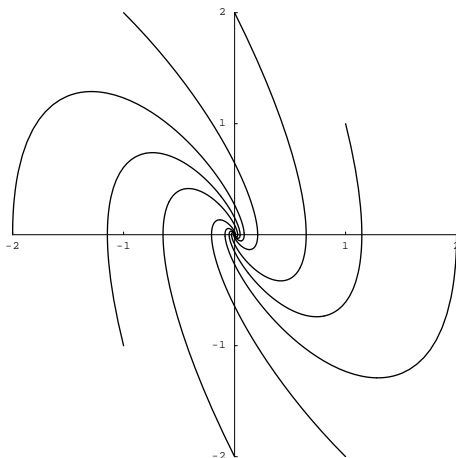
1.862. The first-order system

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & \pi \\ -2\pi & -2\pi \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

induces the flow $Y : (\mathbf{R}^+, 0, +) \rightarrow \text{Set}$ with $Y(*) = \mathbf{R}^2$ and $Y(t) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} x(t, x_0, y_0) \\ y(t, x_0, y_0) \end{pmatrix} = \begin{pmatrix} x_0 \cos(\pi t) + (x_0 + y_0) \sin(\pi t) \\ y_0 \cos(\pi t) - (2x_0 + y_0) \sin(\pi t) \end{pmatrix} e^{-2\pi t}.$$

Trajectories, other than the equilibrium, rotate clockwise around $(0, 0)$ with decaying amplitude.



The function

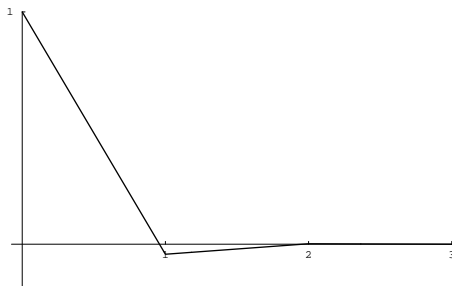
$$\tau(x_0, y_0) = \begin{cases} 0 & \text{if } x_0 = 0; \\ \frac{1}{2} & \text{if } x_0 = -y_0 \text{ and } x_0 \neq 0; \\ -\frac{1}{2} \arctan \frac{x_0}{x_0 + y_0} & \text{if } \frac{x_0}{x_0 + y_0} < 0; \\ \frac{1}{2} \arctan \frac{x_0}{x_0 + y_0} & \text{otherwise} \end{cases}$$

gives the earliest time that the solution having $x(0) = x_0$ and $y(0) = y_0$ crosses the y -axis. It is an example of a Poincaré map [Wi]. Kan extensions can be used to construct a flow from this map and a representation of Y into this new flow [1.863]–[1.864]. I have found no use for such a representation, however.

1.863. Define $X : (\mathbf{N}, 0, +) \rightarrow \text{Set}$ by $X(*) = \mathbf{R}$ and $X(1)(z) = -z e^{-z}$. States of $\text{Ran}(X) : (\mathbf{R}^+, 0, +) \rightarrow \text{Set}$ are certain functions $\mathbf{R}^+ \rightarrow \mathbf{R}$. An example is the continuous φ with graph connecting the points

$$(n, (-1)^n e^{-n}).$$

The flow shifts graphs of such functions to the left.



1.864. Define a function $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ by

$$f \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = y \tau(x_0, y_0), x_0, y_0$$

f assigns to each (x_0, y_0) the position at which the orbit through (x_0, y_0) crosses the y -axis. Calculations in the four cases defining τ show that $f : Y \circ \iota \rightarrow X : (\mathbf{N}, 0, +) \rightarrow \mathbf{Set}$ is a natural transformation:

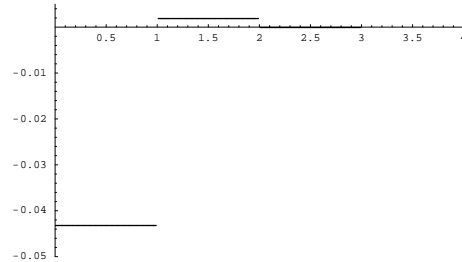
$$\begin{array}{ccc} Y(*) & \xrightarrow{f} & X(*) \\ \mathbf{Y}(1) \downarrow & & \downarrow \mathbf{X}(1) \\ Y(*) & \xrightarrow{f} & X(*) \end{array}$$

\mathbf{j} maps (x_0, y_0) to the function $\mathbf{R}^+ \rightarrow \mathbf{R}$ obtained by applying f to points of the trajectory of (x_0, y_0) .

That is, $\mathbf{j}(x_0, y_0)$ evaluated at t is the y -position at which the orbit through (x_0, y_0) will cross the y -axis once time has already advanced by t along this orbit.

For example,

$$\mathbf{j} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (t) = \begin{cases} 1 & \text{if } t = 0; \\ (-1)^{n+1} e^{-(n+1)} & \text{if } n < t \leq n + 1. \end{cases}$$



1.865. The universal property characterizing left Kan extensions along ι is listed in [3.64].

Theorem 3.643: If \mathcal{C} has coequalizers and ϵ coproducts then \mathcal{C} has a left adjoint.

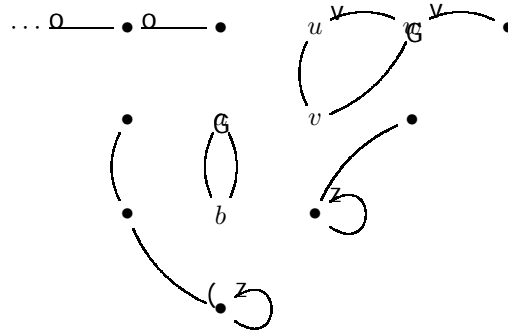
A left adjoint may be computed using the coequalizers and coproducts described in the proof. Formulas with $\mathcal{C} = \mathbf{Set}$ are listed in [3.644].

1.87. From a continuous-time transition $(\mathbf{R}^+, \leq) \rightarrow \mathcal{C}$, the functor \mathcal{C}^0 produces a discrete-time transition $(\mathbf{N}, \leq) \rightarrow \mathcal{C}$. Universal mapping properties characterizing right and left Kan extensions along ι^0 are listed in [3.73], [3.74].

Theorem 3.732: If \mathcal{C} has equalizers and countable products then \mathcal{C}^0 has a right adjoint.

Theorem 3.742: If \mathcal{C} has coequalizers and countable coproducts then \mathcal{C}^0 has a left adjoint.

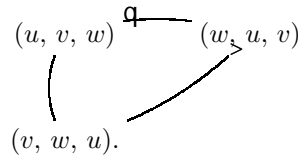
1.88. The functor $\mathcal{C}^{\setminus n} : \mathcal{C}^{(\mathbb{N}_n; 0; +)} \rightarrow \mathcal{C}^{(\mathbb{N}; 0; +)}$ places periodic iterators into the context of all iterators [3.82]. A right Kan extension of $\text{Set}^{\setminus n}$ maps each action to its maximal invariant subobject having period n [page 152 of L6], [3.8]. The image under $\text{Ran}_{\setminus 2}$ of the iterator $(\mathbb{N}, 0, +) \rightarrow \text{Set}$ depicted by



is

$$\begin{array}{c} (a, b) \\ \left(\begin{array}{c} \\ \end{array} \right) \\ (b, a). \end{array}$$

Its image under $\text{Ran}_{\setminus 3}$ is



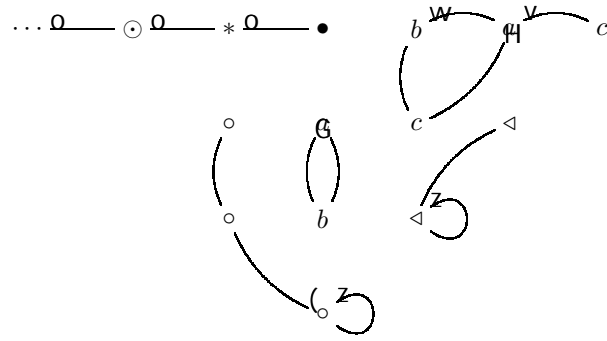
1.881. A right Kan extension along \natural_n of an iterator $X : (\mathbb{N}, 0, +) \rightarrow \mathcal{C}$ is an action $\text{Ran}_{\setminus n}(X) : (\mathbb{N}_n, 0, +) \rightarrow \mathcal{C}$ and equivariant $\varepsilon : \text{Ran}_{\setminus n}(X) \circ \natural_n \Rightarrow X$ with the following properties: for each $Y : (\mathbb{N}_n, 0, +) \rightarrow \mathcal{C}$ and equivariant $f : Y \circ \natural_n \Rightarrow X$, there is a unique $\mathfrak{p} : Y \Rightarrow \text{Ran}_{\setminus n}(X)$ for which $\varepsilon \circ (\mathfrak{p} * \text{id}_{\setminus n}) = f$:

$$\begin{array}{ccc} Y \cdots \cdots \xrightarrow{\mathfrak{p}} \text{Ran}_{\setminus n}(X) & & \begin{array}{c} X \\ \text{wavy arrow} \\ \text{Ran}_{\setminus n}(X) \circ \natural_n \end{array} \\ \mathcal{C}^{(\mathbb{N}_n; 0; +)} & & \mathcal{C}^{(\mathbb{N}; 0; +)} \end{array}$$

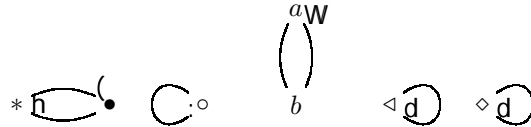
Theorem 3.833: If \mathcal{C} has equalizers and countable products then $\mathcal{C}^{\setminus n}$ has a right adjoint.

A right adjoint may be computed using the equalizers and products described in the proof. Formulas with $\mathcal{C} = \text{Set}$ are listed in [3.834].

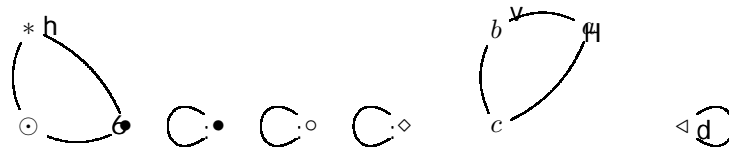
1.882. The image under $\text{Lan}_{\mathbb{N}_2}$ of the iterator $(\mathbb{N}, 0, +) \rightarrow \text{Set}$ depicted by



is

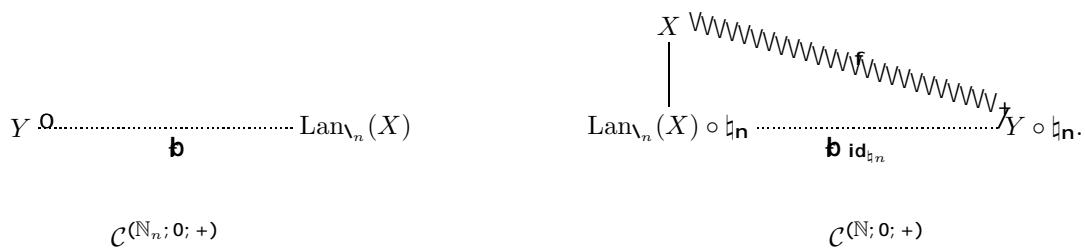


where the symbols indicate the image of η . Its image under $\text{Lan}_{\mathbb{N}_3}$ is



These illustrate the “collapsing of states” described on page 152 of [L6]. From another perspective, $\text{Lan}_{\mathbb{N}_n}$ produces dynamic systems which have **relaxed** [1.22] to a certain periodic condition.

1.883. A left Kan extension along \mathfrak{h}_n of an iterator $X : (\mathbb{N}, 0, +) \rightarrow \mathcal{C}$ is a periodic iterator $\text{Lan}_{\mathbb{N}_n}(X) : (\mathbb{N}_n, 0, +) \rightarrow \mathcal{C}$ and equivariant $\eta : X \Rightarrow \text{Lan}_{\mathbb{N}_n}(X) \circ \mathfrak{h}_n$ with the following property: for each periodic iterator $Y : (\mathbb{N}_n, 0, +) \rightarrow \mathcal{C}$ and equivariant $f : X \Rightarrow Y \circ \mathfrak{h}_n$, there is a unique equivariant $\mathfrak{p} : \text{Lan}_{\mathbb{N}_n}(X) \Rightarrow Y$ for which $(\mathfrak{p} * id_{\mathbb{N}_n}) \circ \eta = f$:



Theorem 3.843: If \mathcal{C} has coequalizers and countable coproducts then $\mathcal{C}^{\mathbb{N}_n}$ has a left adjoint.

A left adjoint may be computed using the coequalizers and coproducts described in the proof. Formulas with $\mathcal{C} = \text{Set}$ are listed in [3.844].

1.89. Stochastic processes are functors into a category Mes studied by Michéle Giry in [G] but defined by Lawvere in a 1965 unpublished manuscript. Let Mes denote the category with measurable spaces as objects and measurable functions as morphisms [III.4]. There is a functor

$$\Pi : \text{Mes} \rightarrow \text{Mes}$$

associating to each space \mathcal{X} its set $\Pi(\mathcal{X})$ of probability measures equipped with the smallest σ -algebra for which the evaluation functions $\text{ev}_{\mathbf{E}} : \Pi(\mathcal{X}) \rightarrow [0, 1]$ are measurable. A measurable $f : \mathcal{X} \rightarrow \mathcal{X}^0$ induces $\Pi(f) : \Pi(\mathcal{X}) \rightarrow \Pi(\mathcal{X}^0)$ by inverse images of events: $\Pi(f)(m)(E^0) = m(f^{-1}(E^0))$.

Mes has measurable spaces as objects. Morphisms $\mathcal{X} \rightarrow \mathcal{Y}$ in Mes are measurable functions $\mathcal{X} \rightarrow \Pi(\mathcal{Y})$ assigning probability measures on \mathcal{Y} to points of \mathcal{X} . For each \mathcal{X} there is a measurable $\eta_{\mathcal{X}} : \mathcal{X} \rightarrow \Pi(\mathcal{X})$ associating Dirac measures to points. $\eta_{\mathcal{X}}$ is the identity morphism for \mathcal{X} . For each \mathcal{X} there is a measurable $\mu_{\mathcal{X}} : \Pi^2(\mathcal{X}) \rightarrow \Pi(\mathcal{X})$ defined by

$$\mu_{\mathcal{X}}(\omega)(E) = \int \text{ev}_{\mathbf{E}} d\omega$$

for ω a probability measure on $\Pi(\mathcal{X})$ and $E \in \mathcal{E}$. Composition of morphisms $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $g : \mathcal{Y} \rightarrow \mathcal{Z}$ in Mes is

$$g \bullet f = \mu_{\mathcal{Z}} \circ \Pi(g) \circ f.$$

1.891. Regarding the category Mes , let $(\Omega, \{q_i\})$ be a limit of a functor $G : \mathcal{C} \rightarrow \text{Mes}$ for which \mathcal{C} is a filtered poset (i.e. a poset with binary joins). Giry described [G] conditions on the projections q_i which guarantee that $(F(\Omega), \{F(q_i)\})$ satisfies certain universal mapping properties in Mes . Edalat [E] showed that a functor category related to Mes has weak pullbacks.

1.892. The functor $\Pi : \text{Mes} \rightarrow \text{Mes}$ is modified to give $\mathfrak{H} : \text{PreMes} \rightarrow \text{PreMes}$ [IV.11] and these two functors are used to define new categories \mathfrak{MPT} and MPT having measure-preserving transformations as morphisms [IV.12], [IV.21]. Since I have not developed any important properties of these categories, their descriptions are relegated to Appendix IV. Objects of MPT , however, are the “fundamental objects of study” in [P]. The latter use of **objects** is not in the categorical sense and [P] does not discuss category theory. Work in [IV.22] leads to the

Conjecture: Caratheodory outer measure induces a functor $\mathfrak{MPT} \rightarrow \text{MPT}$.

1.893. A $n \times m$ matrix is **stochastic** if each entry is nonnegative and each column sum is 1. The category StM having positive integers as objects and stochastic matrices as morphisms is defined in this work [2.1]. Identity morphisms are identity matrices. Composition is matrix multiplication. It is a tool for investigating Mes .

Theorem 2.122: Mes has a subcategory isomorphic to StM .

Some basic properties of StM are developed.

Theorem 2.3: StM has finite, nonempty coproducts.

The proof uses functors $\text{Set} \rightarrow \text{Mes}$ [III.431] and $\text{Mes} \rightarrow \text{Mes}$ [IV.362] which are known to preserve colimits.

Theorem 2.2: StM has finite, weak products.

Proof of this theorem is by direct computation. [2.214] is a sample calculation. Counterexamples establish

Theorem 2.221: StM has neither products nor weak pullbacks.

A result of the work in Chapter 2 is the

Conjecture: Product measures induce weak products in Mes.

A deficiency of Chapter 2 is that it does not have descriptions of monics, epics, or isomorphisms in StM.

1.894. For finite stochastic processes, like the marbles situation described in [1.548], matrix representations facilitate calculations. Transfer of a single marble is a model of the partially ordered set $\bullet \xrightarrow{\quad} \bullet$ in the category StM. Nodes are interpreted as the object $8 \in |\text{StM}|$ since there are eight accessible states. The arrow is interpreted as the stochastic matrix

$$T = \begin{matrix} \text{O} & & & & & & & & & \text{1} \\ \begin{matrix} \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \end{matrix} & \begin{matrix} 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \end{matrix} & \begin{matrix} \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \\ \text{|||||} \end{matrix} \end{matrix}.$$

which is a morphism $8 \xrightarrow{T} 8$ in StM.

If there are lots of marbles, a listing of all states is impractical. One can monitor the system, for example, by counting the marbles in box A. Let $\mathcal{X} = (X, \mathcal{P}(X))$ be the measurable space with $X = \{0, 1, 2, 3\}$. The function $f : S \rightarrow X$ indicating the number of marbles in box A is a random variable[†]. A deterministic process is a special kind of a stochastic one [IV.371] and the measurement f is represented by:

	AAA	AAB	ABA	BAA	ABB	BAB	BBA	BBB
0:	0	0	0	0	0	0	0	1
1:	0	0	0	0	1	1	1	0
2:	0	1	1	1	0	0	0	0
3:	1	0	0	0	0	0	0	0

Each entry gives the probability of obtaining the row-labeling measurement when the system is in the column-labeling state. It may be construed as a morphism $8 \xrightarrow{f} 4$ in StM [2.122].

[†] The measurable space \mathcal{S} was defined in [1.548].

1.895. Although StM has only weak products [2.2], rather than products, the construction of symbolic dynamics for poset actions (nonautonomous systems) [3.341] may be followed to construct a morphism induced by the measurement f . In this application, symbolic dynamics gives information about probabilities of sequences of measurements. Suppose the system were initially in state AAB, for example. The likelihood of obtaining measurement 2 before transferring a marble, then measurement 1 after a transfer is the product of two probabilities: the probability that the initial measurement will be 2, the probability that the measurement will be 1 if the system starts in state AAB and one marble is transferred. These likelihoods are tabulated below and constitute a morphism \mathfrak{p} induced by f .

	<i>AAA</i>	<i>AAB</i>	<i>ABA</i>	<i>BAA</i>	<i>ABB</i>	<i>BAB</i>	<i>BBA</i>	<i>BBB</i>
01:	0	0	0	0	0	0	0	1
10:	0	0	0	0	1/3	1/3	1/3	0
12:	0	0	0	0	2/3	2/3	2/3	0
21:	0	2/3	2/3	2/3	0	0	0	0
23:	0	1/3	1/3	1/3	0	0	0	0
32:	1	0	0	0	0	0	0	0

Moreover, each entry gives the probability of obtaining the row-labeling sequence of measurements if the system is initially in the column-labeling state.

1.896. Successive transitions are modeled by the graph $\bullet \text{---} \diagup \bullet \text{---} \diagdown \bullet$. The corresponding induced \mathfrak{p} is tabulated below:

	<i>AAA</i>	<i>AAB</i>	<i>ABA</i>	<i>BAA</i>	<i>ABB</i>	<i>BAB</i>	<i>BBA</i>	<i>BBB</i>
010:	0	0	0	0	0	0	0	1/3
012:	0	0	0	0	0	0	0	2/3
101:	0	0	0	0	7/27	7/27	7/27	0
103:	0	0	0	0	2/27	2/27	2/27	0
121:	0	0	0	0	14/27	14/27	14/27	0
123:	0	0	0	0	4/27	4/27	4/27	0
210:	0	4/27	4/27	4/27	0	0	0	0
212:	0	14/27	14/27	14/27	0	0	0	0
230:	0	2/27	2/27	2/27	0	0	0	0
232:	0	7/27	7/27	7/27	0	0	0	0
321:	2/3	0	0	0	0	0	0	0
323:	1/3	0	0	0	0	0	0	0

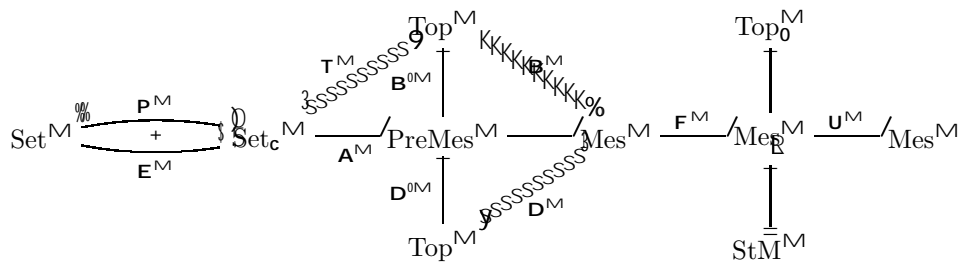
The entry with column label AAB and row label 230, however, is not the probability of obtaining measurement sequence 2-3-0 for a system initially in state AAB. It is the product of three probabilities: the probability that

the initial measurement will be 2 if the system starts in state AAB, the probability that the measurement will be 3 if the system starts in state AAB and one marble is transferred, the probability that the measurement will be 0 if the system starts in state AAB and two marbles are transferred. The measurement sequence 2-3-0 is impossible since only one marble is transferred at a time.

Likelihoods of measurement sequences are tabulated below. These values were not obtained from the construction in [3.341] modified for use with weak products.

	<i>AAA</i>	<i>AAB</i>	<i>ABA</i>	<i>BAA</i>	<i>ABB</i>	<i>BAB</i>	<i>BBA</i>	<i>BBB</i>
010:	0	0	0	0	0	0	0	1/3
012:	0	0	0	0	0	0	0	2/3
121:	0	0	0	0	4/9	4/9	4/9	0
123:	0	0	0	0	2/9	2/9	2/9	0
101:	0	0	0	0	1/3	1/3	1/3	0
210:	0	2/9	2/9	2/9	0	0	0	0
212:	0	4/9	4/9	4/9	0	0	0	0
232:	0	1/3	1/3	1/3	0	0	0	0
321:	2/3	0	0	0	0	0	0	0
323:	1/3	0	0	0	0	0	0	0

1.8(10). Universes of mathematical discourse arising in the context of dynamic systems include: categories of sets [II.1], topological spaces [III.2], and measurable spaces [III.4]; categories composed of binary relations [II.2] and of transition probabilities [IV.3]. Representations among these induce [1.35(10)] representations among categories of dynamic systems:



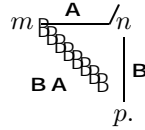
The original contributions of this work are those involving Set_c [II.1] and StM [2.1].

2. The category composed of stochastic matrices

2.1. StM

Call an $n \times m$ real matrix stochastic if it has nonnegative entries and each column sum is 1.

The category StM composed of stochastic matrices has as objects all nonnegative integers. $\text{StM}(m, n)$ is the set of $n \times m$ stochastic matrices. Identity morphisms are identity matrices. Composition is matrix multiplication:



$A_{n \ m}$ indicates that A is a matrix with n rows and m columns. $A_{i,j}$ is the entry in row i and column j .

2.11. StM is a category.

Because: for matrices $A_{n \ m}$ and $B_{p \ n}$ with $\prod_{i=1}^n A_{i,j} = 1$ and $\prod_{e=1}^n B_{e,k} = 1$, $(BA)_{p \ m} = C$ has $C_{u,v} = \prod_{x=1}^n B_{u,x} A_{x,v}$. Since

$$\prod_{u=1}^p C_{u,v} = \prod_{u=1}^p \prod_{x=1}^n B_{u,x} A_{x,v} = \prod_{x=1}^n \prod_{u=1}^p B_{u,x} A_{x,v} = \prod_{x=1}^n A_{x,v} \prod_{u=1}^p B_{u,x} = \prod_{x=1}^n A_{x,v} = 1,$$

BA is stochastic and composition is well-defined. Identity matrices serve as identity morphisms in StM. Matrix multiplication is associative. ■

2.12. Denote the set $\{1, 2, \dots, n\}$ by \underline{n} . Let Mes^S denote the full subcategory of Mes having as objects all $P(\underline{n}) = (\underline{n}, \mathcal{P}(\underline{n}))$ for some positive integer n . The functor $P : \text{Set} \rightarrow \text{Mes}$ was defined in [III.431].

2.121. There is a functor $M : \text{Mes}^S \rightarrow \text{StM}$ with $M(P(\underline{m})) = m$. For $f \in \text{Mes}^S(P(\underline{m}), P(\underline{n}))$, $M(f)$ is $n \times m$ with

$$M(f)_{i,j} = f(j)(\{i\}).$$

Because: $f(j)$ a probability measure on $P(\underline{n})$ implies

$$1 = \sum_{i=1}^n f(j)(\{i\}) = \sum_{i=1}^n M(f)_{i,j}.$$

M maps identity morphisms to identity matrices:

$$M(\eta_{P(\underline{n})})_{i,j} = \eta_{P(\underline{n})}(j)(\{i\}) = \chi_{\text{fig}}(j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in \text{Mes}(P(\underline{n}), P(\underline{p}))$, $\text{ev}_{\text{fkg}}(g(j)) = g(j)(\{k\}) = M(g)_{\mathbf{k}; \mathbf{j}}$ implies

$$\text{ev}_{\text{fkg}} \circ g = \prod_{\mathbf{d}=1}^{\mathbf{X}} M(g)_{\mathbf{k}; \mathbf{d}} \chi_{\text{fdg}}.$$

This justifies equality five of:

$$\begin{aligned} M(g \bullet f)_{\mathbf{k}; \mathbf{j}} &= (g \bullet f)(j)(\{k\}) \\ &= \mu_{\mathbf{P}(\underline{p})} \circ \Pi(g) \circ f(j)(\{k\}) \\ &= \int \text{ev}_{\text{fkg}} d[\Pi(g)(f(j))] \\ &= \int \text{ev}_{\text{fkg}} \circ g d[f(j)] \\ &= \prod_{\mathbf{d}=1}^{\mathbf{X}} M(g)_{\mathbf{k}; \mathbf{d}} f(j)(\{d\}) \\ &= \prod_{\mathbf{d}=1}^{\mathbf{X}} M(g)_{\mathbf{k}; \mathbf{d}} M(f)_{\mathbf{d}; \mathbf{j}} \\ &= (M(g) M(f))_{\mathbf{k}; \mathbf{j}}. \end{aligned}$$

2.122. $M : \text{Mes}^{\mathbf{S}} \rightarrow \text{StM}$ is an isomorphism with inverse $S : \text{StM} \rightarrow \text{Mes}^{\mathbf{S}}$ with $S(m) = P(\underline{m})$. For an $n \times m$ stochastic matrix A

$$S(A)(j)(\{i\}) = A_{i; j} \quad \text{for } j \in \underline{m} \text{ and } i \in \underline{n}.$$

Because: A stochastic implies

$$\prod_{i=1}^{\mathbf{X}} S(A)(j)(\{i\}) = \prod_{i=1}^{\mathbf{X}} A_{i; j} = 1$$

and $A_{i; j} \geq 0$. That is, $S(A)(j)$ is a probability distribution function on $P(\underline{m})$. For $E \in \mathcal{P}(\underline{n})$,

$$S(A)(j)(E) = \prod_{i \in E} S(A)(j)(\{i\}).$$

This gives the additivity needed to conclude that $S(A)(j)$ is a probability measure on $S(n)$. Since $S(m)$ has $\mathcal{P}(\underline{m})$ as its σ -algebra, $S(A) : S(m) \rightarrow S(n)$ is measurable. Calculations in [2.121] show that S preserves identities and composition. ■

2.123. Let $\text{Mes}^{\mathbf{f}}$ denote the full subcategory of Mes having as objects all $P(X) = (X, \mathcal{P}(X))$ with X a nonempty, finite set and let $i : \text{Mes}^{\mathbf{S}} \rightarrow \text{Mes}^{\mathbf{f}}$ be inclusion.

2.124. The composite

$$\text{StM} \xrightarrow[=]{\mathbf{S}} \text{Mes}^{\mathbf{S}} \xrightarrow{\mathbf{i}} \text{Mes}^{\mathbf{f}}$$

is an equivalence.

Because: S an isomorphism and i a full inclusion imply $i \circ S$ a full embedding. The composite has a representative image. Fix $P(X) \in |\mathbf{Mes}^{\mathbf{f}}|$. By the well-ordering principle [Co], X has a linear ordering. Choose one and denote X by $\{x_1, x_2, \dots, x_n\}$. Let

$$f : P(X) \rightarrow P(\underline{n})$$

be $x_i \mapsto i$ and let

$$g : P(\underline{n}) \rightarrow P(X)$$

be $i \mapsto x_i$. These give transition kernels

$$\eta_{P(\underline{n})} \circ f : P(X) \rightarrow \Pi(P(\underline{n})) \quad \text{and} \quad \eta_{P(X)} \circ g : P(\underline{n}) \rightarrow \Pi(P(X))$$

which are isomorphisms. [IV.36] justifies equality three of

$$\begin{aligned} (\eta_{P(X)} \circ g) \bullet (\eta_{P(\underline{n})} \circ f) &= \mu_{P(X)} \circ \Pi(\eta_{P(X)} \circ g) \circ \eta_{P(\underline{n})} \circ f \\ &= \mu_{P(X)} \circ \Pi(\eta_{P(X)}) \circ \Pi(g) \circ \eta_{P(\underline{n})} \circ f \\ &= \Pi(g) \circ \eta_{P(\underline{n})} \circ f \\ &= \eta_{P(X)} \circ g \circ f \\ &= \eta_{P(X)} \end{aligned}$$

and naturality of η to justify equality four. Equalities of the following are justified similarly:

$$\begin{aligned} (\eta_{P(\underline{n})} \circ f) \bullet (\eta_{P(X)} \circ g) &= \mu_{P(\underline{n})} \circ \Pi(\eta_{P(\underline{n})} \circ f) \circ \eta_{P(X)} \circ g \\ &= \mu_{P(\underline{n})} \circ \Pi(\eta_{P(\underline{n})}) \circ \Pi(f) \circ \eta_{P(X)} \circ g \\ &= \Pi(f) \circ \eta_{P(X)} \circ g \\ &= \eta_{P(\underline{n})} \circ f \circ g \\ &= \eta_{P(\underline{n})} : \quad \blacksquare \end{aligned}$$

2.13. There is an embedding $V : \mathbf{StM} \rightarrow \mathbf{Vect}$ via $n \mapsto \mathbf{R}^n$. For $A \in \mathbf{StM}(m, n)$, $V(A)$ is matrix multiplication: $x \in \mathbf{R}^m$ implies $V(A)(x) = Ax$. V is neither full nor faithful.

Because: Distinct $n \times m$ stochastic matrices A and B must have distinct entries at some row-column location, (\hat{i}, \hat{j}) . Let $e_{\hat{j}}$ denote the column vector with m rows, a 1 in row \hat{j} , and zeros elsewhere. $V(A)$ and $V(B)$ are distinct in \mathbf{Vect} because $V(A)e_{\hat{j}} \neq V(B)e_{\hat{j}}$.

V does not reflect isomorphisms: for $A = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 1 \end{pmatrix}$, $V(A)$ is an isomorphism in \mathbf{Vect} with inverse $B = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$. If A had an inverse in \mathbf{StM} it would be B but B is not stochastic.

V is not full since not every matrix is stochastic. \blacksquare

\mathbf{StM} admits nonempty coproducts [2.3] preserved by V . It also has finite, weak products which are not isomorphic to coproducts [2.2]. Since \mathbf{Vect} is linear, V cannot map weak products to products. It maps them to tensor products, however.

2.131. A monic in StM implies $S(A)$ monic in Vect . A epic in StM implies $S(A)$ epic in Vect .

Because: S is an embedding so reflects monics and epics. ■

2.132. $A \in \text{StM}(m, m)$ is monic i epic.

2.133. $m \cong n$ in StM if and only if $m = n$.

Because: $A \in \text{StM}(m, n)$ an isomorphism implies $V(A) \in \text{Vect}(\mathbb{R}^m, \mathbb{R}^n)$ an isomorphism. But $\mathbb{R}^m \cong \mathbb{R}^n$ in Vect if and only if $m = n$. ■

2.14. There is a unique stochastic matrix representing the evolution of a system from a stage with n accessible states to a stage with only one accessible state:

1 is a terminator in StM .

Because: for $m \in |\text{StM}|$, $! = (1 \ 1 \ \dots \ 1)_1 \ m$ is stochastic. It is the unique morphism in $\text{StM}(m, 1)$. ■

Moreover, $!$ is the only subterminator in StM .

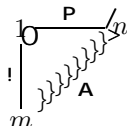
In StM an object n represents a system with a finite number of states x_1, x_2, \dots, x_n . It is also short-hand for the measurable space $P(\underline{n}) = (\underline{n}, \mathcal{P}(\underline{n}))$, an object of $\mathcal{C}\text{Meas}$. Probability measures on this space are recovered

in StM as points of n . $P \in \text{StM}(1, n)$ implies $P = \begin{matrix} \text{B} & p_1 \\ @ & \vdots \\ & \text{A} \\ & p_n \end{matrix}$ with $1 = \sum p_i$.

2.141. A point of an object is necessarily monic, hence, gives a subobject. Moreover,

In StM the subobject of n induced by $P \in \text{StM}(1, n)$ is minimal.

Because: $A = P!$ in



implies

$$A_{r;c} = (P!)_{r;c} = \prod_{k=1}^n P_{r;k} U_{k;c} = P_{r;1} U_{1;c} = P_{r;1}$$

for $1 \leq r \leq n$ and $1 \leq c \leq m$. A has a copy of P as each of its m columns. Moreover,

$$t \xrightarrow{U} /m \xrightarrow{A} /n$$

has

$$(AU)_{r;c} = \prod_{k=1}^n A_{r;k} U_{k;c} = \prod_{k=1}^n P_{r;1} U_{k;c} = P_{r;1} \prod_{k=1}^n U_{k;c} = P_{r;1}$$

for $1 \leq r \leq n$ and $1 \leq c \leq t$ where the last equality holds since U is stochastic. So

$$t \begin{array}{c} \text{U} \\ \oplus \\ \text{V} \end{array} \Big|_n \quad \text{implies} \quad t \begin{array}{c} \text{U} \\ \oplus \\ \text{V} \end{array} \Big|_n \xrightarrow{\text{A}} /_n$$

hence, A monic implies $U = V$. This implies that m is a subterminator, hence, a terminator [2.14]. $A = P$ because the single column of A is P . ■

2.2. Weak products in StM

StM has finite, weak products.

Because: StM has a terminator and any terminator is a weak terminator. Given objects m and n ,

$$m \xrightarrow{0} mn \xrightarrow{0} /_n$$

is a weak product with

$$\begin{aligned} \pi_{r;c} &= \begin{cases} 1 & \text{if } r = \frac{c-1}{n} + 1; \\ 0 & \text{otherwise;} \end{cases} & \text{for } 1 \leq r \leq m \text{ and } 1 \leq c \leq mn \\ \pi_{r;c}^0 &= \begin{cases} 1 & \text{if } c \equiv r \pmod{n}; \\ 0 & \text{otherwise.} \end{cases} & \text{for } 1 \leq r \leq n \text{ and } 1 \leq c \leq mn \end{aligned}$$

where $[[\cdot]]$ denotes the floor function: $[[3.14159]] = 3$, for example. To prove the weak product assertion, consider $A \in \text{StM}(p, m)$ and $B \in \text{StM}(p, n)$:

$$\begin{array}{c} \text{A} \quad \text{B} \\ \text{p} \quad \text{p} \\ \text{m} \quad \text{mn} \quad \text{/}_n \end{array}$$

Define an $mn \times p$ matrix (A, B) as follows. For a positive integer γ let σ denote the γ -cycle $(12 \cdots \gamma)$ in the symmetric group on γ symbols. Then, for example, $\sigma_5(1) = 2$, $\sigma_5^2(1) = 3$, and $\sigma_5^0(1) = 1$. For $1 \leq r \leq mn$ and $1 \leq c \leq p$, let

$$(A, B)_{r;c} = A_{\lfloor (r-1)/n \rfloor + 1; c} B_{r-n+1; c}.$$

(A, B) is a well-defined $mn \times p$ matrix. It is stochastic and makes the following commute:

$$\begin{array}{c} \text{A} \quad \text{B} \\ \text{p} \quad \text{p} \\ \text{m} \quad \text{mn} \quad \text{/}_n \\ \text{(A, B)} \end{array}$$

Each entry of (A, B) is a product of nonnegative quantities, hence, is itself nonnegative. For $1 \leq c \leq p$,

$$\begin{aligned} \prod_{r=1}^{mn} (A, B)_{r;c} &= \prod_{r=1}^{mn} A_{\lfloor (r-1)/n \rfloor + 1; c} B_{r-n+1; c} \\ &= \prod_{\alpha=0}^{m-1} \prod_{\beta=0}^{n-1} A_{\alpha+1; c} B_{\alpha n + \beta + 1; c} \\ &= \prod_{\alpha=0}^{m-1} A_{\alpha+1; c} \prod_{\beta=0}^{n-1} B_{\beta+1; c} \\ &= \prod_{\alpha=0}^{m-1} A_{\alpha+1; c} \prod_{\beta=0}^{n-1} B_{\beta+1; c} = \prod_{\alpha=0}^{m-1} A_{\alpha+1; c} = 1 \end{aligned}$$

where the last two equalities are justified by the fact that A and B are stochastic. Thus, (A, B) is stochastic.

For $1 \leq r \leq m$ and $1 \leq c \leq p$,

$$\begin{aligned}
 [\pi(A, B)]_{r;c} &= \sum_{k=1}^n \pi_{r;k}(A, B)_{k;c} \\
 &= \sum_{k=1}^n \delta_{r;[(k-1)=n]+1} A_{\frac{[(k-1)/n]}{m}}(1;c) B_{\frac{k-1}{n}}(1;c) \\
 &= \sum_{=0}^n \delta_{r;[(\alpha+n-1)=n]+1} A_{\frac{[(\alpha+n-1)/n]}{m}}(1;c) B_{\frac{\alpha+n-1}{n}}(1;c) \\
 &= \sum_{=0}^n \delta_{r;[\alpha+(n-1)=n]+1} A_{\frac{[(\alpha+(n-1)/n]}{m}}(1;c) B_{\frac{\beta-1}{n}}(1;c) \\
 &= \sum_{=0}^n \delta_{r;\alpha+1} A_{\frac{\alpha}{m}}(1;c) B_{\frac{\beta-1}{n}}(1;c) \\
 &= \sum_{=1}^n A_{\frac{\alpha}{m}}(1;c) B_{\frac{\beta-1}{n}}(1;c) = A_{r;c} \sum_{=1}^n B_{\frac{\beta-1}{n}}(1;c) = A_{r;c}.
 \end{aligned}$$

For $1 \leq r \leq n$ and $1 \leq c \leq p$,

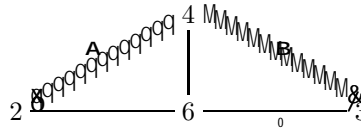
$$\begin{aligned}
 [\pi^0(A, B)]_{r;c} &= \sum_{k=1}^n \pi_{r;k}^0(A, B)_{k;c} \\
 &= \sum_{=0}^n \pi_{r;\alpha+n}^0(A, B)_{\alpha+n;c} \\
 &= \sum_{=0}^n \pi_{r;\alpha+n}^0 A_{\frac{[(\alpha+n-1)/n]}{m}}(1;c) B_{\frac{\alpha+n-1}{n}}(1;c) \\
 &= \sum_{=0}^n \pi_{r;\alpha+n}^0 A_{\frac{[(\alpha+(n-1)/n]}{m}}(1;c) B_{\frac{\beta-1}{n}}(1;c) \\
 &= B_{r;c} \sum_{=0}^n \pi_{r;\alpha+n}^0 A_{\frac{\alpha+[(\beta-1)/n]}{m}}(1;c) \\
 &= B_{r;c} \sum_{=0}^n \pi_{r;\alpha+n}^0 A_{\frac{\alpha}{m}}(1;c) = B_{r;c} \sum_{=0}^n A_{\frac{\alpha}{m}}(1;c) = B_{r;c}.
 \end{aligned}$$

Hence, StM has weak products.

Uniqueness of the induced map does not hold: let $A \in \text{StM}(4, 2)$ be $\begin{matrix} 1/2 & 2/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 3/4 & 4/5 \end{matrix}$ and let $B \in \text{StM}(4, 3)$ be $\begin{matrix} 1 & 0 & 1/3 & 0 \\ 0 & 1/2 & 1/3 & 1 \\ 0 & 1/2 & 1/3 & 0 \end{matrix}$ \mathbf{A} ; then $\pi = \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{matrix}$ and $\pi^0 = \begin{matrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{matrix}$ \mathbf{A} . The two StM-morphisms

$$(A, B) = \begin{matrix} \mathbf{O} & 1/2 & 0 & 1/12 & 0 & \mathbf{1} \\ \mathbf{m} & 0 & 1/3 & 1/12 & 1/5 & \mathbf{C} \\ \mathbf{m} & 0 & 1/3 & 1/12 & 0 & \mathbf{C} \\ \mathbf{m} & 1/2 & 0 & 1/4 & 0 & \mathbf{A} \\ \mathbf{m} & 0 & 1/6 & 1/4 & 4/5 & \mathbf{A} \\ \mathbf{m} & 0 & 1/6 & 1/4 & 0 & \mathbf{A} \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbf{O} & 1/2 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{m} & 0 & 1/2 & 0 & 1/5 & \mathbf{C} \\ \mathbf{m} & 0 & 1/6 & 1/4 & 0 & \mathbf{C} \\ \mathbf{m} & 1/2 & 0 & 1/3 & 0 & \mathbf{A} \\ \mathbf{m} & 0 & 0 & 1/3 & 4/5 & \mathbf{A} \\ \mathbf{m} & 0 & 1/3 & 1/12 & 0 & \mathbf{A} \end{matrix}$$

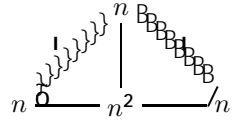
both make



commute

■

2.21. For any $n \in |\text{StM}|$, the diagonal $\Delta \in \text{StM}(n, n^2)$ is (I, I) where I denotes the $n \times n$ identity matrix:



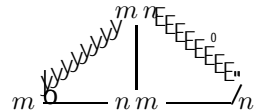
For $1 \leq r \leq n^2$ and $1 \leq c \leq n$,

$$\begin{aligned} \Delta_{r;c} &= I_{\lfloor (r-1)/n \rfloor}^{[\lfloor (r-1)/n \rfloor]}(\mathbf{1}; \mathbf{c}) I_{n-r-1}^{r-1}(\mathbf{1}; \mathbf{c}) \\ &= I_{\lfloor (\alpha+\beta-1)/n \rfloor}^{[\lfloor (\alpha+\beta-1)/n \rfloor]}(\mathbf{1}; \mathbf{c}) I_{n-\alpha-\beta-1}^{\alpha+\beta-1}(\mathbf{1}; \mathbf{c}) \quad \text{where } r = \alpha n + \beta, 0 \leq \alpha \leq n-1, \text{ and } 1 \leq \beta \leq n \\ &= I_{\alpha}^{\alpha}(\mathbf{1}; \mathbf{c}) I_{\beta-1}^{\beta-1}(\mathbf{1}; \mathbf{c}) \\ &= I_{+1; \mathbf{c}} I_{-1; \mathbf{c}} \\ &= \begin{cases} \mathbf{1} & \text{if } r = (c-1)n + c; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

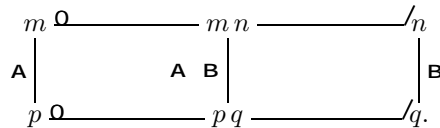
$$\begin{matrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{matrix}$$

For $n = 2$, the diagonal morphism is $\begin{matrix} \mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{C} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A} \\ \mathbf{0} & \mathbf{1} & & \end{matrix}$.

2.211. For $m, n \in |\text{StM}|$, the twist morphism $\zeta \in \text{StM}(mn, nm)$ is defined by

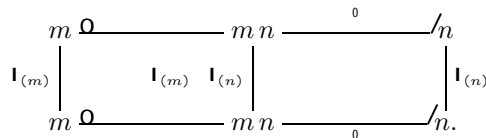


2.212. A category with weak products of objects has weak products of morphisms. The weak product of $A \in \text{StM}(m, p)$ and $B \in \text{StM}(n, q)$ is defined by



2.213. Weak products induce a functor $\times : \text{StM} \times \text{StM} \rightarrow \text{StM}$.

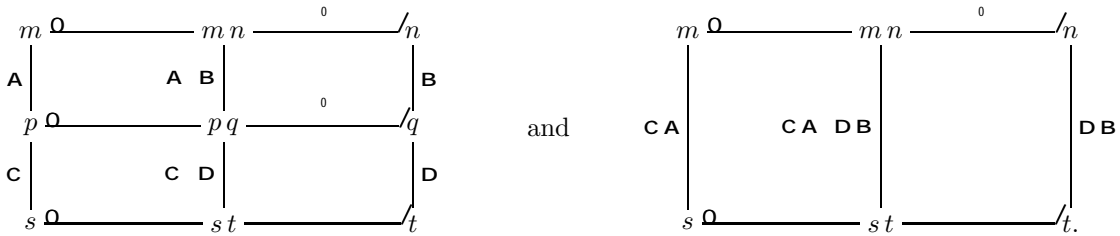
Because: a weak product of identity morphisms is an identity. Consider



For $1 \leq r \leq mn$ and $1 \leq c \leq mn$, let $r = \alpha n + \beta$ where $0 \leq \alpha \leq m - 1$ and $1 \leq \beta \leq n$ and let $c = \gamma n + \delta$ where $0 \leq \gamma \leq m - 1$ and $1 \leq \delta \leq n$,

$$\begin{aligned}
 (I(m) \times I(n))_{r;c} &= (I(m), I(n))_{r;c} \\
 &= (I(m) \pi)_{\frac{\alpha}{m}(1);c} (I(n) \pi^0)_{\frac{\beta}{n}-1;c} \\
 &= (I(m) \pi)_{+1;c} (I(n) \pi^0)_{!};c \\
 &= \prod_{u=1}^{\alpha} I_{+1;!} \pi_{!};c \prod_{v=1}^{\beta-1} I_{!} \pi_{!}^0;c \\
 &= \pi_{+1;c} \pi^0;c \\
 &= \pi_{+1; n+} \pi^0_{; n+} \\
 &= \begin{cases} \pi_{+1; n+} & \text{if } \delta = \beta; \\ 0 & \text{otherwise} \end{cases} \\
 &= \begin{cases} 1 & \text{if } \delta = \beta \text{ and } \gamma = \alpha; \\ 0 & \text{otherwise} \end{cases} \\
 &= \begin{cases} 1 & \text{if } r = c; \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

Formation of weak products preserves composites. Consider



For $1 \leq r \leq st$ and $1 \leq c \leq mn$, let $r = \lambda t + \delta$ where $0 \leq \lambda \leq s - 1$ and $1 \leq \delta \leq t$ and let $c = \alpha n + \beta$ where $0 \leq \alpha \leq m - 1$ and $1 \leq \beta \leq n$. Then

$$\begin{aligned}
 (CA \times DB)_{r;c} &= ((CA \pi), (DB \pi^0))_{r;c} \\
 &= (CA \pi)_{\frac{\lambda}{s}((r-1)/t+1)(1);c} (DB \pi^0)_{\frac{\delta}{t}-1(1);c} \\
 &= ((CA \pi)_{\frac{\lambda}{s}(1);c} ((DB \pi^0)_{\frac{\delta}{t}-1(1);c} \\
 &= \prod_{u=1}^{\lambda} (CA)_{\frac{\lambda}{s}(1);u} \pi_u;c \prod_{v=1}^{\delta-1} (DB)_{\frac{\delta}{t}-1(1);v} \pi_v^0;c \\
 &= \prod_{u=1}^{\lambda} (CA)_{\frac{\lambda}{s}(1);u} \pi_u; n+ \prod_{v=1}^{\delta-1} (DB)_{\frac{\delta}{t}-1(1);v} \pi_v^0; n+ \\
 &= (CA)_{\frac{\lambda}{s}(1); +1} (DB)_{\frac{\delta}{t}-1(1);} \\
 &= (CA)_{+1; +1} (DB)_{;}.
 \end{aligned}$$

This must be the same as:

$$\begin{aligned}
((C \times D)(A \times B))_{r;c} &= \prod_{j=1}^q (C \times D)_{r;j} (A \times B)_{j;c} \\
&= \prod_{j=1}^q (D \gamma, D \gamma^0)_{r;j} (A \pi, B \pi^0)_{j;c} \\
&= \prod_{j=1}^q (C \gamma)_{\frac{[(r-1)/q]+1}{s};j} (D \gamma^0)_{\frac{r-1}{t};j} (A \pi)_{\frac{[(j-1)/q]+1}{p};c} (B \pi^0)_{\frac{j-1}{q};c} \\
&= \prod_{j=1}^q (C \gamma)_{\frac{\lambda}{s};j} (D \gamma^0)_{\frac{\delta}{t};j} (A \pi)_{\frac{[(j-1)/q]+1}{p};c} (B, \pi^0)_{\frac{j-1}{q};c} \\
&= \prod_{j=1}^q (C \gamma)_{+1;j} (D, \gamma^0)_{;j} (A, \pi)_{\frac{[(j-1)/q]+1}{p};c} (B, \pi^0)_{\frac{j-1}{q};c} \\
&= \prod_{i=0}^{q-1} (C \gamma)_{+1; q+} (D, \gamma^0)_{; q+} (A \pi)_{\frac{i}{p};c} (B \pi^0)_{\frac{i}{q};c} \\
&= \prod_{i=0}^{q-1} (C, \gamma)_{+1; q+} (D, \gamma^0)_{; q+} (A, \pi)_{+1;c} (B \pi^0)_{;c} .
\end{aligned}$$

Note that

$$\begin{aligned}
(B \pi^0)_{;c} &= \prod_{i=1}^q B_{;i} \pi^0_{i;c} = \prod_{i=1}^q B_{;i} \pi^0_{i; n+} = B_{; n+} \\
(A \pi)_{+1;c} &= \prod_{i=1}^q A_{+1;i} \pi_{i;c} = \prod_{i=1}^q A_{+1;i} \pi_{i; n+} = A_{+1; n+} \\
(C \gamma)_{+1; q+} &= \prod_{i=1}^q C_{+1;i} \gamma_{i; q+} = C_{+1; q+} \\
(D \gamma^0)_{; q+} &= \prod_{i=1}^q D_{;i} \gamma^0_{i; q+} = D_{; q+} .
\end{aligned}$$

From these it follows that

$$\begin{aligned}
((C \times D)(A \times B))_{r;c} &= \prod_{i=0}^{q-1} C_{+1; q+} D_{; q+} A_{+1; q+} B_{; q+} \\
&= \prod_{i=0}^{q-1} C_{+1; q+} A_{+1; q+} D_{; q+} B_{; q+} \\
&= \prod_{i=0}^{q-1} C_{+1; q+} A_{+1; q+} D_{; q+} B_{; q+} \\
&= (C A)_{+1; q+} (D B)_{; q+} .
\end{aligned}$$

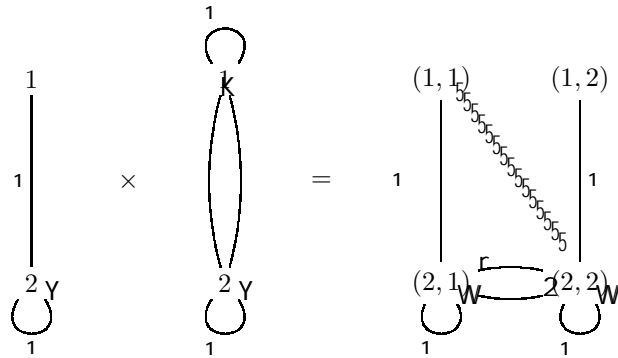
2.214. Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{pmatrix}$ with $\alpha, \beta \in [0, 1]$. Then

$$A \times B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1-\alpha & \beta & 1-\alpha & \beta & 0 \\ \alpha & 1-\beta & \alpha & 1-\beta & 0 \end{pmatrix} .$$

States in the product have pairs of labels:

$$1 = (1, 1), \quad 2 = (1, 2), \quad 3 = (2, 1), \quad 4 = (2, 2)$$

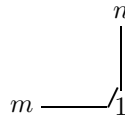
The example may be illustrated by labeled graphs.



Results of this section suggest that Mes has weak products but not products.

2.22. StM does not admit all pullbacks.

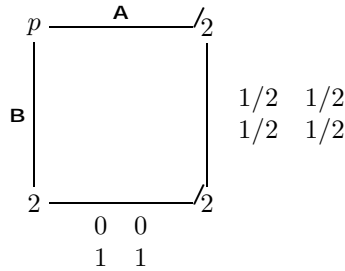
Because: a pullback of



is a product of m and n .

2.221. Since StM admits weak products, it admits some weak pullbacks. However, StM **does not admit all weak pullbacks**.

Because: Commutativity of



with $A = \begin{matrix} \alpha_1 & \cdots & \alpha_p \\ 1 - \alpha_1 & \cdots & 1 - \alpha_p \end{matrix}$ and $B = \begin{matrix} \beta_1 & \cdots & \beta_p \\ 1 - \beta_1 & \cdots & 1 - \beta_p \end{matrix}$ implies

$$\begin{matrix} 1/2 & \cdots & 1/2 \\ 1/2 & \cdots & 1/2 \end{matrix} = \begin{matrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{matrix} \quad A = \begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix} \quad B = \begin{matrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \end{matrix} .$$

2.3. Coproducts in StM

StM has finite, nonempty coproducts.

Because: for objects m and n of StM, $m + n$ is also an object. In Set,

$$\underline{m} \xrightarrow{\quad} / \underline{m+n} \mathbf{0} \xrightarrow{\quad} \underline{n}$$

is a coproduct with $\lambda(i) = i$ for $1 \leq i \leq m$ and $\lambda^0(j) = j + m$ for $1 \leq j \leq n$. $P : \text{Set} \rightarrow \text{Mes}$ a left adjoint [III.431] implies preservation of colimits, hence,

$$P(\underline{m}) \xrightarrow{\quad} / P(\underline{m+n}) \mathbf{0} \xrightarrow{\quad} P(\underline{n})$$

is a coproduct in Mes. $\text{Mes} \rightarrow \text{Mes}$ via $(X, \mathcal{M}) \rightarrow (X, \mathcal{M})$ and $f \mapsto \eta_{\mathfrak{r}} \circ f$ a left adjoint [IV.362] implies that

$$P(\underline{m}) \xrightarrow{\lambda^{\square}} / P(\underline{m+n}) \mathbf{0} \xrightarrow{\lambda^0 \square} P(\underline{n})$$

is a coproduct in $\text{Mes}^{\mathfrak{s}}$. As $\text{Mes}^{\mathfrak{s}}$ is a full subcategory of Mes , the previous diagram also describes a $\text{Mes}^{\mathfrak{s}}$ -coproduct. The isomorphism $M : \text{Mes}^{\mathfrak{s}} \rightarrow \text{StM}$ gives a coproduct in StM:

$$m \xrightarrow{\mathbf{M}(\lambda^{\square})} / \underline{m+n} \mathbf{0} \xrightarrow{\mathbf{M}(\lambda^0 \square)} p.$$

■

We may use the definitions of η , λ , and λ^0 to unravel the construction described in the above proof. Let $\tilde{\lambda} = M(\eta \circ \lambda)$ and $\tilde{\lambda}^0 = M(\eta \circ \lambda^0)$. For $1 \leq r \leq m+n$ and $1 \leq c \leq m$,

$$\tilde{\lambda}_{\mathfrak{r};c} = \eta(\lambda(c))(\{r\}) = \eta(c)(\{r\}) = \begin{cases} 1 & \text{if } r = c; \\ 0 & \text{otherwise.} \end{cases}$$

For $1 \leq r \leq m+n$ and $1 \leq c \leq n$,

$$\tilde{\lambda}^0_{\mathfrak{r};c} = \eta \circ (\lambda^0(c))(\{r\}) = \eta \circ (c+m)(\{r\}) = \begin{cases} 1 & \text{if } r = c+m; \\ 0 & \text{otherwise.} \end{cases}$$

A coproduct in StM may, thus, be denoted by

$$m \xrightarrow{\begin{matrix} I_m \\ 0_{\mathfrak{n} \ \mathfrak{m}} \end{matrix}} / \underline{m+n} \mathbf{0} \xrightarrow{\begin{matrix} 0_{\mathfrak{m} \ \mathfrak{n}} \\ I_n \end{matrix}} n.$$

Given $A \in \text{StM}(m, p)$ and $B \in \text{StM}(n, p)$, the induced map $(A \mid B) \in \text{StM}(m+n, p)$

$$\begin{array}{ccc} & I_m & 0_{\mathfrak{m} \ \mathfrak{n}} \\ & 0_{\mathfrak{n} \ \mathfrak{m}} & I_n \\ m & \xrightarrow{\quad} & / \underline{m+n} \mathbf{0} \xrightarrow{\quad} n \\ & \text{PPPPPPPP} & \text{PPPPPPPP} \\ & \text{PPPPPPPP} & \text{PPPPPPPP} \\ & \text{A} & \text{B} \\ & \downarrow & \\ & & p \end{array}$$

may be computed[†]. For $1 \leq r \leq p$ and $1 \leq c \leq m$,

$$A_{r;c} = \sum_{k=1}^{m+n} (A | B)_{r;k} \tilde{\lambda}_{k;c} = (A | B)_{r;c}.$$

For $1 \leq r \leq p$ and $m+1 \leq c \leq m+n$,

$$B_{r;c} = \sum_{k=1}^{m+n} (A | B)_{r;k} \tilde{\lambda}_{k;c}^0 = (A | B)_{r;c}.$$

2.31. The coproduct of 2 and 3 in StM is

$$\begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \text{mmmm} & \text{CC} & \text{CC} \\ @ & \text{A} & \text{A} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array} \quad \begin{array}{ccc} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \text{mmmm} & \text{CC} & \text{CC} \\ @ & \text{A} & \text{A} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{array}$$

2 ————— / 5 ————— 3

2.311. For any $n \in |\text{StM}|$, the codiagonal $\nabla \in \text{StM}(2n, n)$ is $(I | I)$ where I denotes the $n \times n$ identity matrix:

$$\begin{array}{ccc} I & & 0_{n \ n} \\ 0_{n \ n} & / & 0 \\ n & \text{oooooooooooooooooooo} & n \end{array}$$

For $n = 3$, $\nabla \in \text{StM}(6, 3)$ is $\begin{array}{ccc} \mathbf{0} & & \mathbf{1} \\ \text{mmmm} & \text{CC} & \text{CC} \\ @ & \text{A} & \text{A} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}$.

2.312. For $m, n \in |\text{StM}|$, the twist morphism $\zeta \in \text{StM}(m+n, n+m)$ is $\begin{array}{ccc} 0_{n \ m} & & I \\ I & & 0_{m \ n} \end{array}$:

$$\begin{array}{ccc} I & & 0_{m \ n} \\ 0_{n \ m} & / & 0 \\ m & \text{pppppppppppppppppppp} & m+n \\ 0_{n \ m} & & I \\ I & & 0_{m \ n} \end{array}$$

For $m = 2$ and $n = 3$, $\zeta \in \text{StM}(5, 5)$ is $\begin{array}{ccc} \mathbf{0} & & \mathbf{1} \\ \text{mmmm} & \text{CC} & \text{CC} \\ @ & \text{A} & \text{A} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{array}$. This particular twist morphism is idempotent with $\zeta^5 = I_{(5)}$

[†] The standard notation for the map from a coproduct induced by A and B is $\begin{array}{c} A \\ B \end{array}$. The notation $(A | B)$ will be used when working in StM since it suggests how the induced matrix is formed from the matrices A and B .

2.313. A category with binary coproducts of objects has coproducts of morphisms. The coproduct of $A \in \text{StM}(m, p)$ and $B \in \text{StM}(n, q)$ is defined by

$$\begin{array}{ccc}
 & I & \\
 & \text{0}_n \text{ m} & \text{0}_m \text{ n} \\
 m & \xrightarrow{\quad} & m+n \\
 \text{A} \downarrow & & \downarrow \text{A+B} \\
 p & \xrightarrow{\quad} & p+q \\
 & I & \\
 & \text{0}_q \text{ p} & \text{0}_p \text{ q} \\
 & & I
 \end{array}$$

where $A + B = \begin{matrix} A & \text{0}_p \text{ n} \\ \text{0}_q \text{ m} & B \end{matrix}$.

For $A = \begin{matrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \end{matrix} \in \text{StM}(3, 2)$ and $B = \begin{matrix} 1/3 & 0 & 1/2 \\ 1/3 & 1 & 0 \\ 1/3 & 0 & 1/2 \end{matrix} \in \text{StM}(3, 3)$,

$$A + B = \begin{matrix} \text{O} & & & & & \text{1} \\ 1 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 1/2 \\ 0 & 0 & 0 & 1/3 & 1 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 1/2 \end{matrix}$$

2.314. Binary coproducts on a category, \mathcal{C} , induce a functor $\oplus : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$.

Because: binary coproducts always induce such a functor. ■